RESEARCH STATEMENT – ROBERT SCHNEIDER

My primary research interest is in systematizing number theory at the intersection of its additive and multiplicative branches, and proving applications in partitions, q-series, modular forms, analytic number theory, combinatorics and the physical sciences. Much like the natural numbers $\mathbb{N}$, the set $\mathcal{P}$ of partitions teems with patterns and relations. Now, Alladi-Erdős point out in [2] the prime decompositions of natural numbers are in bijective correspondence with the set of partitions into prime parts, if we associate 1 to the empty partition $\emptyset$. Might some number-theoretic theorems arise as images in $\mathbb{N}$ (i.e., in prime partitions) of greater algebraic and set-theoretic structures in $\mathcal{P}$? As a member of Emory's Working Group on Number Theory and Molecular Simulation, I use partition theory plus statistical physics to model phenomena in chemistry. To what extent are theorems in number theory part of the tapestry of nature?

In the 1970s, Andrews developed a beautiful theory of *partition ideals* [3] using ideas from lattice theory, to unify classical results on partition bijections and generating functions, and discover whole families of new bijections—teasing the possibility of a universal algebra of partitions. Hoping to build on Andrews's theory, and charged up by results of Alladi, Euler, Fine, Ono, Ramanujan, Zagier and other authors, in my Ph.D. research I look for further evidence of algebraic structures in the partitions. It turns out many objects from multiplicative and analytic number theory are special cases of large-scale partition-theoretic structures. Here I survey my publications in partition theory and q-series [7, 8, 9, 11, 12, 13], quantum modular forms [10, 14] and theoretical chemistry [17]. I omit some publications in elementary number theory, music theory and history of mathematics; I also study prime distribution, but have no papers yet.

1. Intersection of additive and multiplicative number theory

My first published mathematics paper [7] was a collaboration with Marie Jameson. We proved a number of new partition relations applying Möbius inversion and other methods from multiplicative number theory to give combinatorial interpretations for coefficients of certain interrelated q-series. Mixing the Möbius function with q-series sparked my curiosity. I rediscovered a partition analog of $\mu(n)$ studied privately by Alladi, which led me to a multiplicative theory of partitions that contains the classical cases.

For brevity we assume common partition theory and q-series concepts. For an integer partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r), \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 1$, we let "$\lambda_i \in \lambda$" mean $\lambda_i \in \mathbb{N}$ is a part of $\lambda \in \mathcal{P}$, and "$\lambda \vdash n$" mean λ is a partition of $n \geq 0$, with the convention $\emptyset \vdash 0$. We use the *size* $|\lambda|$ (sum of parts), *length* $\ell(\lambda)$ (number of parts), and define a nonstandard statistic, the "integer" of a partition (introduced in [11]).

Definition 1. *We define the integer of λ, notated n_λ, to be the product of the parts:*

$$n_\lambda := \lambda_1 \lambda_2 \cdots \lambda_r$$

We assume the convention $n_\emptyset := 1$ (it is an empty product). Pushing further in the multiplicative direction, in [12] I define a simple, intuitive multiplication operation on the elements of $\mathcal{P}$ (equivalent to the operation $\oplus$ in Andrews's theory), as well as division.

Definition 2. *We define the product $\lambda \lambda'$ of two partitions $\lambda, \lambda' \in \mathcal{P}$ as the multi-set union of their parts listed in weakly decreasing order, e.g. $(5, 2, 2)(6, 5, 1) = (6, 5, 5, 2, 2, 1)$. The empty partition $\emptyset$ serves as the multiplicative identity. We say a partition δ divides (or is a "subpartition" of) λ and write $\delta | \lambda$, if all the parts of δ are also parts of λ, including multiplicity. When $\delta | \lambda$ we might discuss the quotient $\lambda / \delta \in \mathcal{P}$ formed by deleting the parts of δ from λ.*

We can now discuss the partition-theoretic analog of $\mu(n)$ mentioned above.

Definition 3. *For $\lambda \in \mathcal{P}$ we define a partition-theoretic Möbius function $\mu_\mathcal{P}(\lambda)$ as follows:*

$$\mu_\mathcal{P}(\lambda) := \begin{cases} 0 & \text{if } \lambda \text{ has any part repeated,} \\ (-1)^{\ell(\lambda)} & \text{otherwise} \end{cases}$$

Note that if λ is a prime partition, $\mu_\mathcal{P}(\lambda)$ reduces to $\mu(n_\lambda)$. Just as in the classical case, we have by inclusion-exclusion the following, familiar relation.

Proposition 4 (Proposition 3.2 in [12]). *Summing $\mu_{\mathcal{P}}(\delta)$ over the subpartitions δ of $\lambda \in \mathcal{P}$ gives*

$$\sum_{\delta | \lambda} \mu_{\mathcal{P}}(\delta) = \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

I also prove a partition-theoretic generalization of Möbius inversion in [12]. Now, the classical Möbius function has a close companion in the Euler phi function $\varphi(n)$, and $\mu_{\mathcal{P}}$ has a companion as well.

Definition 5. *For $\lambda \in \mathcal{P}$ we define a partition-theoretic phi function*

$$\varphi_{\mathcal{P}}(\lambda) := n_{\lambda} \prod_{\substack{\lambda_i \in \lambda \\ \text{without} \\ \text{repetition}}} (1 - \lambda_i^{-1}).$$

Clearly $\varphi_{\mathcal{P}}(\lambda)$ reduces to $\varphi(n_{\lambda})$ if λ is a prime partition, and, as with $\mu_{\mathcal{P}}$, generalizes classical results.

Proposition 6 (Propositions 3.6 and 3.7 in [12]). *We have that*

$$\sum_{\delta | \lambda} \varphi_{\mathcal{P}}(\delta) = n_{\lambda}, \qquad \varphi_{\mathcal{P}}(\lambda) = n_{\lambda} \sum_{\delta | \lambda} \frac{\mu_{\mathcal{P}}(\delta)}{n_{\delta}}.$$

There are generalizations of other arithmetic objects in partition theory (see [12]), as well as connections to group theory that I am currently pursuing. I plan to continue to work toward an algebra of partitions.

2. Combinatorial applications in q-series

Now we outline a few applications of the above multiplicative ideas. For $f : \mathcal{P} \to \mathbb{C}$ define

$$(1) \qquad\qquad\qquad F(\lambda) := \sum_{\delta | \lambda} f(\delta).$$

As I prove in [12], sums over subpartitions interact in a very surprising and beautiful way with the q-Pochhammer symbol $(q; q)_{\infty} := \prod_{n \geq 1}(1 - q^n), |q| < 1$. Here is one general result.

Proposition 7 (Theorems 4.1 and 4.9 of [12]). *With the above notation, we have the relations*

$$(q; q)_{\infty} \sum_{\lambda \in \mathcal{P}} F(\lambda) q^{\lambda} = \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{\lambda}, \qquad \frac{1}{(q; q)_{\infty}} \sum_{\lambda \in \mathcal{P}} F(\lambda) q^{\lambda} = \sum_{\lambda \in \mathcal{P}} \widetilde{F}(\lambda) q^{\lambda}$$

where $\widetilde{F}(\lambda) = \sum_{\delta | \lambda} F(\delta)$. Moreover, partition Möbius inversion gives $f(\lambda) = \sum_{\delta | \lambda} F(\delta) \mu_{\mathcal{P}}(\lambda / \delta)$.

In effect, multiplying by $(q; q)_{\infty}$ "undoes" subpartition sums in partition-indexed q-series, and dividing by $(q; q)_{\infty}$ sends the coefficients into these sums. In [12] I make somewhat clever use of Proposition 7 to give explicit formulas for coefficients of the *q-bracket of Bloch-Okounkov* (see [5, 15]), an operator from statistical physics connected to quasi-modular and p-adic modular forms, and to address other q-series. Here is one application from [12]; recall the Jacobi triple product $j(z; q) := (z; q)_{\infty}(z^{-1}q; q)_{\infty}(q; q)_{\infty}$.

Proposition 8 (Corollary 6.2 of [12]). *For $z \neq 1$ the reciprocal of the triple product is given by*

$$\frac{1}{j(z; q)} = \sum_{n \geq 0} c_n q^n \quad \text{with} \quad c_n = c_n(z) = (1 - z)^{-1} \sum_{\lambda \vdash n} \sum_{\delta | \lambda} \sum_{\varepsilon | \delta} z^{\text{crk}(\varepsilon)},$$

where $\text{crk}(*)$ denotes the crank of a partition as defined by Andrews-Garvan (see [4]).

In [13], I apply ideas from [11, 12] to relate $j(z; q)$ and the q-bracket to the odd-order universal mock theta function $g_3(z, q)$ of Gordon-McIntosh (see [6]) and the rank generating function $\widetilde{U}_k(q)$ (resp. $U_k(q)$) for unimodal (resp. strongly unimodal) sequences with k-fold peak, proving formulas such as these.

Proposition 9 (Corollaries 4 and 8 of [13]). *For $|q| < 1 < |z|$, we have*

$$g_3(z^{-1}, q^{-1}) = \frac{z}{1 - z} \sum_{k=1}^{\infty} \widetilde{U}_k(z, q) z^{-k} q^k.$$

For $|z| < 1$, the radial limit as $q \to \zeta_m$ an mth order root of unity is given by

$$g_3(z, \zeta_m) = \frac{z-1}{z} \sum_{k=1}^{\infty} U_k(-z, \zeta_m) z^k \zeta_m^{-k}.$$

As-yet-unpublished generalizations of Proposition 7 (which I omit for space) yield further examples, like formulas for the coefficients of $(q;q)_\infty^k$ for any k. Let $m_i(\lambda)$ be the multiplicity of i as a part of λ.

Proposition 10. *Ramanujan's tau function can be written*

$$\tau(n) = \sum_{\lambda \vdash (n-1)} (-1)^{\ell(\lambda)} \binom{24}{m_1(\lambda)} \binom{24}{m_2(\lambda)} \binom{24}{m_3(\lambda)} \cdots.$$

Proposition 11. *The number $P_k(n)$ of k-color partitions of n can be written*

$$P_k(n) = \sum_{\lambda \vdash n} \binom{k + m_1(\lambda) - 1}{m_1(\lambda)} \binom{k + m_2(\lambda) - 1}{m_2(\lambda)} \binom{k + m_3(\lambda) - 1}{m_3(\lambda)} \cdots.$$

In the future, I would like to study how structures and symmetries in the partitions might be connected to modularity in q-series, given the combinatorics well known to be encoded in the coefficients.

3. Partition zeta functions

Arithmetic functions and divisor sums are not the only multiplicative phenomena with connections in partition theory. In [11] and with co-authors Ken Ono and Larry Rolen in [8], I investigate a broad class of *partition zeta functions* (and in [8], partition Dirichlet series) arising from a fusion of Euler's product formulas for both the partition generating function and the Riemann zeta function, which admit interesting structure laws and evaluations as well as classical specializations.

Definition 12. *On analogy with the Riemann zeta function $\zeta(s)$, for a subset $\mathcal{P}'$ of $\mathcal{P}$ and value $s \in \mathbb{C}$ for which the series converges, we define a partition zeta function $\zeta_{\mathcal{P}'}(s)$ by*

$$\zeta_{\mathcal{P}'}(s) := \sum_{\lambda \in \mathcal{P}'} n_\lambda^{-s}.$$

If we let $\mathcal{P}'$ equal the partitions $\mathcal{P}_{\mathbb{X}}$ whose parts all lie in some subset $\mathbb{X} \subset \mathbb{N}$, there is also an Euler product

$$\zeta_{\mathcal{P}_{\mathbb{X}}}(s) = \prod_{n \in \mathbb{X}} (1 - n^{-s})^{-1}.$$

Of course, $\zeta(s)$ is the case $\mathbb{X} = \mathbb{P}$; and many classical zeta function identities generalize to partition identities. For instance, for appropriate $s \in \mathbb{C}$, $\mathbb{X} \subset \mathbb{N}$, we easily get familiar-looking relations like these.

Proposition 13. *Just as in the classical cases, we have the following identities:*

$$\frac{1}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)} = \sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \mu_{\mathcal{P}}(\lambda) n_\lambda^{-s}, \qquad \frac{\zeta_{\mathcal{P}_{\mathbb{X}}}(s-1)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)} = \sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \varphi_{\mathcal{P}}(\lambda) n_\lambda^{-s}$$

In [11] I show partition zeta sums over other proper subsets of $\mathcal{P}$ can yield nice closed-form results. To see how subsets influence the evaluations, fix $s = 2$ and sum over three unrelated subsets of $\mathcal{P}$: partitions $\mathcal{P}_{even}$ into even parts, partitions $\mathcal{P}_{prime}$ into prime parts, and partitions $\mathcal{P}_{dist}$ into distinct parts.

Proposition 14 (Corollaries 2.1 and 2.10 in [11])**.** *We have the identities*

$$\zeta_{\mathcal{P}_{even}}(2) = \frac{\pi}{2}, \qquad \zeta_{\mathcal{P}_{prime}}(2) = \frac{\pi^2}{6}, \qquad \zeta_{\mathcal{P}_{dist}}(2) = \frac{\sinh\pi}{\pi}.$$

Notice how different choices of partition subsets induce very different partition zeta values for fixed s. Interestingly, differing powers of π appear in all three examples. Another curious formula involving π arises if we take $s = 3$ and sum on partitions $\mathcal{P}_{\geq 2}$ with all parts ≥ 2 (that is, no parts equal to 1).

Proposition 15 (Proposition 2.3 in [11])**.** *We have that*

$$\zeta_{\mathcal{P}_{\geq 2}}(3) = \frac{3\pi}{\cosh\left(\frac{1}{2}\pi\sqrt{3}\right)}.$$

These formulas are appealing, but they are a little too motley to comprise a *family* like Euler's values

$$\zeta(2k) \in \mathbb{Q}\pi^{2k}.$$

However, in [11] I produce a class of partition zeta functions that does yield nice evaluations like this.

Definition 16. *We define*

$$\zeta_{\mathcal{P}}(\{s\}^k) := \sum_{\ell(\lambda)=k} n_\lambda^{-s},$$

where the sum is taken over all partitions of fixed length $k \geq 1$ (the $k = 1$ case is just $\zeta(s)$).

Proposition 17 (Corollary 2.4 in [11]). *For $s = 2, k \geq 1$, we have the identity*

$$\zeta_{\mathcal{P}}(\{2\}^k) = \frac{2^{2k-1} - 1}{2^{2k-2}}\zeta(2k).$$

For example, we give the following values:

$$\zeta_{\mathcal{P}}(\{2\}^2) = \frac{7\pi^4}{360}, \quad \zeta_{\mathcal{P}}(\{2\}^3) = \frac{31\pi^6}{15120}, \quad \cdots \quad, \quad \zeta_{\mathcal{P}}(\{2\}^{13}) = \frac{22076500342261\pi^{26}}{93067260259985915904000000}, \quad \cdots$$

There are increasingly complicated identities for $\zeta_{\mathcal{P}}(\{2^t\}^k), t \geq 1$, in [11]. My co-authors and I prove more in [8], as follows (which can also be deduced from work of Hoffman).

Proposition 18 (Corollary 5 in [8]). *For $m > 0$ even, we have*

$$\zeta_{\mathcal{P}}(\{m\}^k) \in \mathbb{Q}\pi^{mk}.$$

So these zeta sums over partitions of fixed length really do form a family like Euler's zeta values. Inspired by work of Chamberland-Straub, in [8] we also evaluate partition zeta functions over partitions $\mathcal{P}_{a+m\mathbb{N}}$ whose parts are all $\equiv a$ modulo m. Let Γ denote the usual gamma function, and let $e(x) := e^{2\pi ix}$.

Proposition 19 (Theorem 1 in [8]). *For $n \geq 2$, we have*

$$\zeta_{\mathcal{P}_{a+m\mathbb{N}}}(n) = \Gamma(1 + a/m)^{-n} \prod_{r=0}^{n-1} \Gamma\left(1 + \frac{a - e(r/n)}{m}\right).$$

In [8] we address analytic continuation of certain partition zeta functions, which is somewhat rare. Now, the analytic continuation of $\zeta_{\mathcal{P}}(\{s\}^k)$ is given in [11] for fixed length $k = 2$; we can write

$$(2) \qquad\qquad \zeta_{\mathcal{P}}(\{s\}^2) = \frac{\zeta(2s) + \zeta(s)^2}{2},$$

thus $\zeta_{\mathcal{P}}(\{s\}^2)$ inherits analytic continuation from the Riemann zeta functions on the right. I would like to find further examples of analytic continuation of $\zeta_{\mathcal{P}}(\{s\}^k)$ and other zeta forms. Also, I wish to make rigorous a certain heuristic for locating imaginary parts of the zeros of partition zeta functions.

4. PARTITION FORMULAS FOR ARITHMETIC DENSITIES

Alladi proves in [1] a surprising *duality principle* connecting arithmetic functions to sums over smallest or largest prime factors of divisors, and applies this principle to prove for $\gcd(r, t) = 1$ that

$$(3) \qquad\qquad -\sum_{\substack{n \geq 2 \\ p_{\min}(n) \equiv r(\mathrm{mod}\ t)}} \mu(n)n^{-1} = \frac{1}{\varphi(t)},$$

where $p_{\min}(n)$ denotes the smallest prime factor of n, and $1/\varphi(t)$ represents the proportion of primes in a fixed arithmetic progression modulo t. Using analogous dualities from partition generating functions (smallest/largest parts instead), in [9] my co-authors Ken Ono, Ian Wagner and I extend Alladi's ideas.

Proposition 20 (Theorems 1.3–1.4 of [9]). *For suitable subsets $\mathcal{S}$ of $\mathbb{N}$ with arithmetic density $d_{\mathcal{S}}$,*

$$-\lim_{q \to 1} \sum_{\substack{\lambda \in \mathcal{P} \\ \mathrm{sm}(\lambda) \in \mathcal{S}}} \mu_{\mathcal{P}}(\lambda)q^{|\lambda|} = d_{\mathcal{S}},$$

where $\mathrm{sm}(\lambda)$ denotes the smallest part of λ, and $q \to 1$ from within the unit circle.

In particular, if we denote kth-power-free integers by $\mathcal{S}_{\mathrm{fr}}^{(k)}$, we prove a partition formula to compute $1/\zeta(k)$ as the limiting value of a partition-theoretic q-series as $q \to 1$.

Proposition 21 (Corollary 1.5 of [9])**.** *If $k \geq 2$, then*

$$-\lim_{q \to 1} \sum_{\substack{\lambda \in \mathcal{P} \\ \mathrm{sm}(\lambda) \in \mathcal{S}_{\mathrm{fr}}^{(k)}}} \mu_{\mathcal{P}}(\lambda) q^{|\lambda|} = \frac{1}{\zeta(k)}.$$

Alladi's duality is fully generalized in partition theory; he and I have discussed further applications.

5. Other topics: quantum modular forms, statistical physics, chemistry, probability

Inspired by Zagier's work [16] with Kontsevich's "strange" function $F(q)$, in [10] my co-author Larry Rolen and I construct a vector-valued quantum modular form $\phi(x) := \begin{pmatrix} \theta_1^S(e^{2\pi ix}) & \theta_2^S(e^{2\pi ix}) & \theta_3^S(e^{2\pi ix}) \end{pmatrix}^T$ whose components $\theta_i^S : \mathbb{Q} \to \mathbb{C}$ are similarly "strange". (I study other "strange" series in [14].)

Proposition 22 (Theorem 1.1 of [10])**.** *We have that $\phi(x)$ is a weight $3/2$ vector-valued quantum modular form. In particular, we have that*

$$\phi(z+1) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_{12} \\ 0 & \zeta_{24} & 0 \end{pmatrix} \phi(z) = 0,$$

and we also have

$$\left(\frac{z}{-i}\right)^{-3/2} \phi(-1/z) + \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \phi(z) = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} g(z),$$

where $g(z)$ is a 3-dimensional vector of smooth functions defined in [10] as period integrals.

I note above my work on q-brackets [12, 13], which represent expected values in systems indexed by partitions. In James Kindt's computational chemistry group at Emory, we compute expected values using partition variants of Faà di Bruno's formula, and aim to prove the Law of Mass Action from partition-theoretic first principles. Results of our fast PEACH algorithm for molecular simulation are in [17]. I am also currently applying vector-valued mock modular forms to birth-death processes in probability theory.

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