INFINITE DIMENSIONAL LIE GROUPS WITH APPLICATIONS TO MATHEMATICAL PHYSICS

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Lecture 1 : Infinite dimensional Lie groups. General definitions, finite versus infinite dim. Lie groups, examples, applications of abelian gauge groups to Maxwell's equations, loop groups

Lecture 2: Diffeomorphism groups. Algebraic structure, geometric structure, Lie group structure, Lie algebra, exponential map, ILH Lie groups, gauge groups

Lecture 3: Subgroups of diffeomorphism groups and applications. Volume preserving diffeos (fluid dynamics), symplectomorphisms (plasma physics), contact transformations , global hamiltonian vector fields, quantomorphisms (geom. quantization), gauge transformations (quantum field theory)

Lecture 4: Lie group of Fourier integral operators. Basic definitions and properties. Exact sequence of groups, local section, smoothness of manifold structure, smoothness of group operations, ILH Lie group structure .

Lecture 5: Non-compact manifolds. Applications: Topological Euler equations, non-homogeneous Euler equations. Pseudodifferential operators and quantization.

LECTURE 1

INFINITE DIMENSIONAL LIE GROUPS

LIE GROUPS

In physics as symmetry groups or configuration spaces of dynamical systems.

Finite dimensional examples:

a) **Linear and Angular Momentum:**
Groups of translations and rotations.

b) **Rigid body:** $f(x,t)$ position at time t . Smooth motion : $f(x,t) = A(t) \cdot x$,
$A(t) \in SO(3)$ proper rotations.
$SO(3)$ configuration space and symmetry group.

c) **Heavy top:** Configuration space SO(3), symmetry group circle group S^1, rotations about direction of gravity.
"Eliminating" S^1 symmetry : Euclidean group E_3 rigid motions.

Infinite dimensional examples:

d) **Incompressible fluid:** Configuration space: $Diff_{vol}(\Omega)$ volume preserving diffeomorphisms of Ω. Infinite dimensional Lie group .
$Diff_{vol}(\Omega)$ also symmetry group.

e) **Compressible fluid:** Configuration space: $Diff(\Omega)$ Symmetry group $Diff_\rho(\Omega)$ density ρ preserving diffeomorphisms .

f) **Maxwell-Vlasov Equation:** Configuration space $Sym(\mathbf{R}^6)$, infinite dim. Lie group of canonical transformations.

h) **Maxwell's , Yang-Mills Equations.** Group of gauge transformations , infinite dimensional Lie group .

i) **Soliton equations:** Coadjoint orbits of group of pseudodifferential operator, group of Fourier integral operators.

Basic Definitions and Properties

An infinite dimensional **Lie group** $\mathcal{G}$ is a group and an infinite dimensional manifold with . The 2 structures are compatible , i.e. the group operations are smooth

$$\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G} : \mu(g,h) = g \cdot h \quad \textit{multiplication} \quad C^\infty$$

$$\nu : \mathcal{G} \to \mathcal{G} : \nu(g) = g^{-1} \quad \textit{inversion} \quad C^\infty$$

(implicit function theorem !)

$\mathcal{G}$ is locally diffeomorphic to an ∞ dim. vector space

- Banach space , norm $\|.\|$,
- Hilbert space , inner product $< .,. >$,
- Frechet space , metric $d(.,.)$, no norm

If $\mathcal{G}$ is locally diffeomorphic to $\mathbf{R}^n$,$n < \infty$,
then $\mathcal{G}$ is a finite dimensional Lie group

Examples.

1) V = Banach space: abelian Lie group
$\mu : V \times V \to V, \quad \mu(x,y) = x + y$ and
$\nu : V \to V, \quad \nu(x) = -x.$

2) $GL(n, \mathbf{R})$ linear isomorphisms of $\mathbf{R}^n$
(invertible $n \times n$ matrices),
μ = matrix multiplication
ν = matrix inversion

3) V = Banach space . $GL(V,V)$ invertible elements of $L(V,V)$ = bounded linear operators $A : V \to V$. $\mu(f,g) = f \circ g$, $\nu(f) = f^{-1}$

Differentiable structure of $\mathcal{G}$

Left- & right translations , $g \in \mathcal{G}$:
$L_g : \mathcal{G} \to \mathcal{G}$; $L_g(h) = gh$ diffeomorphism
$R_g : \mathcal{G} \to \mathcal{G}$; $R_g(h) = hg$ diffeomorphism
$(L_g)^{-1} = L_{g^{-1}}, \quad (R_g)^{-1} = R_{g^{-1}}$

Let (U, ϕ) chart at $e \in \mathcal{G}$,
define chart (U_g, ϕ_g) at $g \in \mathcal{G}$

$$U_g = L_g(U) = \{L_g h \mid h \in U\}$$

$$\phi_g = \phi \circ L_{g^{-1}} : U_g \to V, h \mapsto \phi(g^{-1}h)$$

The Lie algebra $\mathbf{g}$ of a Lie group $\mathcal{G}$:

Lie algebra $\mathbf{g} = \{$ left invariant vector fields on $\mathcal{G}\}$
$\simeq T_e\mathcal{G}$ tangent space at the identity

A Vector field X on $\mathcal{G}$ is **left invariant** iff
$(L_g)_* X \ = \ X \ \Leftrightarrow X \in \mathbf{X}_L(\mathcal{G})$
if $X, Y \ \in \ \mathbf{X}_L(\mathcal{G})$, then
$(L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y] = [X,Y]$,
i.e. $[X,Y] \in \mathbf{X}_L(\mathcal{G})$.

Hence $\mathbf{X}_L(\mathcal{G})$ is a Lie subalgebra of $\mathbf{X}(\mathcal{G})$.

The **Lie algebra** $\mathbf{g}$ of $\mathcal{G}$ is defined as $\mathbf{g} = \mathbf{X}_L(\mathcal{G})$

$\xi \in T_e\mathcal{G} \mapsto X_\xi(g) := T_eL_g(\xi) \in T_g\mathcal{G}$ left invariant
and $X_\xi(e) = \xi$
Lie bracket in $T_e\mathcal{G}$, $\xi, \eta \in \mathbf{g}$

$$[\xi, \eta] := [X_\xi, X_\eta](e)$$

- bilinear: $[t\xi_1 + s\xi_2, \eta] = t[\xi_1, \eta] + s[\xi_2, \eta]$
- skew symmetric: $[\xi, \eta] = -[\eta, \xi]$
- Jacobi identity $[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0$

exponential map: $exp : \mathbf{g} \to \mathcal{G}$ defined as follows:
1) $\xi \in \mathbf{g} \mapsto X_\xi$ left invariant vector field.
2) let $\varphi_\xi(t)$ be flow of X_ξ
3) define $exp(\xi) := \varphi_\xi(1)$ local diffeomorphis if $\mathcal{G}$ finite dimensional

Examples

CLASSICAL LIE GROUPS

Vector group: $\mathcal{G} = V$ vector space
$\mu(x, y) = x + y,\ \nu(x) = -x,\ e = 0$, abelian

Lie algebra: $T_eV \simeq V$
left invariant vector field: $u \in T_eV$
$X_u(v) = u, \forall v \in V$ const. Hence $\mathrm{g} = V$ with trivial Lie bracket $[u, v] = 0$ abelian
exponential map $exp : V \to V,\ exp = id$.

MATRIX GROUPS

General linear group $GL(n, \mathbf{R})$

$$\mathcal{G} = GL(n, \mathbf{R}) = \{A \in L(\mathbf{R}^n, \mathbf{R}^n) \mid \det A \neq 0\}$$

invertible $n \times n$ matrices.
det : $L(\mathbf{R}^n, \mathbf{R}^n) \to \mathbf{R}$, continuous, and
$GL(n, \mathbf{R}) = \det^{-1}(\mathbf{R} - \{0\}) \subset L(\mathbf{R}^n, \mathbf{R}^n)$ open submanifold , disconnected, $dim = n^2$

$\mu(A, B) = AB$, matrix multiplication , C^∞ as restriction of continuous bilinear map
$(A, B) \in L(\mathbf{R}^n, \mathbf{R}^n) \times L(\mathbf{R}^n, \mathbf{R}^n) \to AB \in L(\mathbf{R}^n, \mathbf{R}^n)$

$\nu(A) = A^{-1}$ matrix inversion, C^∞ by implicit function theorem $\mu(A, \nu(A)) = e = I$ identity

Lie algebra $\quad \mathrm{g} = L(\mathbf{R}^n, \mathbf{R}^n)$

Lie bracket $[A, B] = AB - BA$

left invariant vector fields on $GL(n, \mathbf{R})$:

$A \in L(\mathbf{R}^n, \mathbf{R}^n), \quad X_A : GL(n, \mathbf{R}) \to L(\mathbf{R}^n, \mathbf{R}^n)$, def by $X_A(Y) = YA$ is left invariant (linear), indeed

$X_A(L_Z Y) = X_A(ZY) = ZYA = (T_Y L_Z) X_A(Y)$

hence the Lie bracket is

$[A, B] = [X_A, X_B](I) = DX_B(I)X_A(I) - DX_A(I)X_B(I)$

$X_B(Z)$ is linear, so $DX_B(I)Z = ZB$ and

$DX_B(I)X_A(I) = AB$ hence $\quad [A, B] = AB - BA$

exponential map: $A \in L(\mathbf{R}^n, \mathbf{R}^n)$ the curve $\gamma_A : \mathbf{R} \to GL(n, \mathbf{R}) : \gamma_A(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i$ is a one-parameter subgroup, $\gamma_A(0) = I$

$\dot{\gamma_A}(t) = \sum_{i=1}^{\infty} \frac{t^{i-1}}{(i-1)!} A^i = \gamma_A(t) A$

Hence γ_A is (unique) integral curve of X_A

exponential map

$exp : L(\mathbf{R}^n, \mathbf{R}^n) \to GL(n, \mathbf{R}) : exp(A) = \gamma_A(1) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$, i.e. $exp(A) = e^A$.

Lie subgroups of $GL(n, \mathbf{R})$

Special linear group $SL(n, \mathbf{R})$

$SL(n, \mathbf{R}) = \{A \in GL(n, \mathbf{R}) | \det A = 1\} = \ det^{-1}\{1\}$.

closed Lie subgroup of $GL(n, \mathbf{R})$, non-compact, connected (2 comp), $dim\ SL(n, \mathbf{R}) = n^2 - 1$

Lie algebra

$$sl(n, \mathbf{R}) = \{A \in L(\mathbf{R}^n, \mathbf{R}^n) |\ trA = 0\}$$

Orthogonal group $O(n)$

$$O(n) = \{A \in\ L(\mathbf{R}^n, \mathbf{R}^n) |\ < Ax, Ay > = < x, y >\}$$

$$\Leftrightarrow AA^T = I\ ,\ detA = \pm 1$$

compact , disconnected (2 comp),
$dim\ O(n) = n(n-1)/2$

Lie algebra

$$o(n, \mathbf{R}) = \{A \in L(\mathbf{R}^n, \mathbf{R}^n) |\ A \text{ skew symmetric}\}$$

Special orthogonal group $SO(n)$

$$\begin{aligned} SO(n) &= \{A \in L(\mathbf{R}^n, \mathbf{R}^n) |\ detA = +1\} \\ &= O(n) \cap SL(n, \mathbf{R}) = comp_I O(n) \end{aligned}$$

compact, connected , $dim\ SO(n) = \frac{1}{2}n(n-1)$

Lie algebra

$$\begin{aligned} so(n, \mathbf{R}) &= \{A \in L(\mathbf{R}^n, \mathbf{R}^n) |\ A \text{ skew symmetric}\} \\ &= o(n) \end{aligned}$$

Symplectic group $Sp(2n)$

$Sp(2n, \mathbf{R}) = \{A \in L(\mathbf{R}^{2n}, \mathbf{R}^{2n}) | A^T J A = J\}$.

$J = \begin{pmatrix} O & I \\ -I & 0 \end{pmatrix}$, noncompact, $dim\ Sp(2n) = 2n^2 + n$

Lie algebra

$$sp(2n, \mathbf{R}) = \{A \in L(\mathbf{R}^n, \mathbf{R}^n) |\ A^T J + JA = 0\}$$

Similar for complex matrix groups $GL(n, \mathbf{C})$

Classical results in finite dimensions which are NOT true in infinite dimensions:

NO Implicit Function Theorem !

1) Exponential map:
$exp : \mathbf{g} \to G : \xi \in \mathbf{g} \mapsto X_\xi$ left invariant vector field with flow $\varphi_\xi(t)$, then $exp(\xi) = \varphi_\xi(1)$ is a local diffemomorphism $\Rightarrow$ canonical coordinates

2) If $f_1, f_2 : G \to H$ are smooth Lie group homomorphisms (G connected) $f_i(gh) = f_i(g) \cdot f_i(h)$, $i = 1, 2$, with $T_e f_1 = T_e f_2$, then $f_1 = f_2$ locally

3) If $f : G \to H$ is a continuous group homomorphism then f is smooth

4) If G is a Lie group and $H \subset G$ a closed subgroup then H is a Lie subgroup (Lie group and submanifold)

5) If G is a Lie group with Lie algebra $\mathbf{g}$ and $\mathbf{h} \subset \mathbf{g}$ is a subalgebra, then there exists a unique connected Lie subgroup $H \subset G$ with $\mathbf{h}$ as its Lie algebra

6) If $\mathbf{g}$ is any finite dim. Lie algebra. There exists a connected Lie group G such that $\mathbf{g} \simeq T_e G$

Infinite dimensional examples

1. Abelian gauge groups

M finite dimensional manifold
$\mathcal{G} = C^\infty(M),\ \mu(f,g) = f+g,\ \nu(f) = -f,\ e = 0$
$\mathbf{g} = T_eC^\infty(M) \simeq C^\infty(M),\ [\xi,\eta] = 0$, abelian
$exp = id : C^\infty(M) \to C^\infty(M)$
C^∞-Frechet Lie group (vector group)
norm-complete:
$C^k(M)$ -norm, $k < \infty \Rightarrow$ Banach Lie group
H^s-Sobolev norm, $s > \frac{1}{2} dim M \Rightarrow$ Hilbert Lie group

Application: Maxwell's equations

E, B electric and magnetic fields

$$\boxed{\dot{E} = curl\ B,\ \dot{B} = -curl\ E,\ div\ B = 0,\ div\ E = \rho}$$

$V =$ vector fields (potentials) on $\mathbf{R}^3$
$P = T^*V = V \times V^* \ni (A, E)$, phase space
L^2 pairing $< A, E > = \int A(x)E(x)dx$,

canonical Poisson bracket for $F, H : P \to \mathbf{R}$

$$\{F,H\}(A,E) = \int \left(\frac{\delta F}{\delta A}\frac{\delta H}{\delta E} - \frac{\delta H}{\delta A}\frac{\delta F}{\delta E}\right)dx$$

Hamiltonian (energy)

$H(A,E) = \frac{1}{2}\int(|B|^2 + |E|^2)dx$

Hamiltons equations $\boxed{\quad \dot{A} = \frac{\delta H}{\delta E}\ , \quad \dot{E} = -\frac{\delta H}{\delta A}}$

let $B = -\ curl\ A$, then $divB = -div\ curlA = 0$
Hamiltons equations $\dot{A} = \frac{\delta H}{\delta E} = E \Rightarrow \dot{B} = -curl\ E$
$\dot{E} = -\frac{\delta H}{\delta A} = -curl\ curlA = curlB$
$divE = \rho$ from symmetry

Gauge invariance: $\mathcal{G} = C^\infty(\mathbf{R}^3)$ acts on V by
$\varphi \cdot A = A + \nabla\varphi$
lifted action to $V \times V^*$, $\varphi \cdot (A,E) = (A + \nabla\varphi, E)$
H is $\mathcal{G}$ invariant
momentum map $J : V \times V^* \to \mathbf{g}^* \simeq$ charge densities
$J(A,E) = div\ E$
reduced phase space for $\rho \in \mathbf{g}^*$
$P_\rho = J^{-1}(\rho)/G = \{(E,B)|div\ E = \rho,\ div\ B = 0\}$

reduced Hamiltonian:

$$H_\rho(E,B) = \frac{1}{2}\int(|E|^2 + |B|^2)dx$$

reduced Poisson bracket:

$$\{F,H\}_\rho(E,B) = \int \left(\frac{\delta F}{\delta E}\cdot curl\ \frac{\delta H}{\delta B} - \frac{\delta H}{\delta E}\cdot curl \frac{\delta F}{\delta B}\right) dx$$

Hamilton's equations on reduced phase space

$$\dot{F} = \{F, H_\rho\}_\rho \Leftrightarrow \begin{cases} \dot{E} = curl\ B \quad , \quad \dot{B} = -curl\ E \\ div\ B = 0 \quad , \quad div\ E = \rho \end{cases}$$

Maxwell's equations

2. Loop groups

Let M be a finite dimensional manifold
$\mathcal{G} = C^\infty(M, \mathbf{R} - \{0\})$
$\mu(f,g) = f \cdot g,\ \nu(f) = f^{-1}\ ,\ e = 1$

$C^k(M, \mathbf{R} - \{0\})$ open in $C^\infty(M, \mathbf{R})$
M compact $\Rightarrow$ Banach Lie group, $k < \infty$
M compact $\Rightarrow H^s(M, \mathbf{R} - \{0\})$, H^s Sobolev class, closed under multiplication if $s > \frac{1}{2} dim M$
Hilbert Lie group

generalize $\mathbf{R} - \{0\}$ to G finite dimensional Lie group
$\mathcal{G} = C^k(M, G),\ \mu(f,g)(x) = f(x) \cdot g(x)$ pointwise
$\nu(f)(x) = f^{-1}(x)$ pointwise
Lie algebra: $\mathbf{g} = C^k(M, g),\ [\xi, \eta](x) = [\xi(x), \eta(x)]$ pointwise

exponential map:
$EXP : \mathbf{g} = C^k(M, g) \to \mathcal{G} = C^k(M, G)$
$EXP(\xi) = exp \circ \xi$ local diffeomorphism

Applications:
Gauge theories, quantum field theory

Special case: $M = S^1$ circle

$\mathcal{G} = C^k(S^1, G) = L^k(G)$ loop group
$\mathbf{g} = C^k(S^1, g) = l^k(g)$ loop algebra

Applications: affine Lie algebras, Kac-Moody Lie algebras (central extensions)
- completely integrable systems
- soliton equations (Toda, KdV, KP)
- quantum field theory