

LECTURE 2

DIFFEOMORPHISM GROUPS

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Let M be a smooth manifold, and let

$$Diff(M) = \{f : M \rightarrow M \mid f \text{ diffeomorphism}\}$$

Algebra structure of $Diff(M)$?

$Diff(M)$ is a group with composition as group operation

$$\mu : Diff(M) \times Diff(M) \rightarrow Diff(M), \mu(f, g) = f \circ g$$

and inversion

$$\nu : Diff(M) \rightarrow Diff(M), \nu(f) = f^{-1}$$

the unit element is $e = id : M \rightarrow M$.

$Diff(M)$ is the **diffeomorphism group** of M .

cohomology, isomorphy, simplicity...

(Banyaga, 70's, 1997)

Geometric structure of $Diff(M)$?

topology, manifold structure, global analysis

Lie group structure of $Diff(M)$?

- C^k , $k < \infty$ diffeomorphisms \Rightarrow Banach manifold (Palais 68, Omori 74)
- C^∞ diffeomorphisms \Rightarrow Frechet manifold (Omori, Gutknecht, Schmid 78)
- H^s Sobolev H^s diffeomorphisms, f is H^s iff in any local chart all derivatives of f up to order s are square integrable, $s > \frac{1}{2} \dim M \Rightarrow$ Hilbert manifold (Ebin-Marsden-Fischer 70, Ratiu-Schmid 79-, Eichhorn Schmid 90's -)

Theorem: Lie group structure of $Diff^s(M)$
 $s > \frac{1}{2} \dim M$

multiplication is C^k

$$\mu : Diff^{s+k}(M) \times Diff^s(M) \rightarrow Diff^s(M)$$

$k = 0 \Rightarrow$ multiplication is continuous

inversion is C^k

$$\nu : Diff^{s+k}(M) \rightarrow Diff^s(M)$$

$k = 0 \Rightarrow$ inversion is continuous

$Diff^\infty(M) = \varprojlim_s Diff^s(M)$ "Frechet Lie group"

Inverse Limit of Hilbert Lie group
 called **ILH Lie group**

The geometric structure of $Diff^s(M)$

$Diff(M) \subset C(M, M)$ open subset
construction of a manifold structures on
 $C^\infty(M, M)$, $C^k(M, M)$, $C^s(M, M)$?

Manifold of maps $C^s(M, N)$

let M be compact, N finite dim. manifolds

$$C(M, N) = \{f : M \rightarrow N \mid f \text{ differentiable}, C^\infty, C^k, H^s\}$$

parameter space \cong tangent space

tangent vectors at a point $f \in C(M, N)$ given by a

curve $\gamma : I \subset \mathbf{R} \rightarrow C(M, N)$, $\gamma(0) = f$, then

$$\dot{\gamma}(0) = \left. \frac{d}{dt} \right|_{t=0} \gamma(t) \in T_f C(M, N), \quad \dot{\gamma}(0) = ?$$

$$x \in M : \gamma_x : I \subset \mathbf{R} \rightarrow N : \gamma_x(t) = \gamma(t)(x)$$

$$\gamma_x(0) = f(x) \Rightarrow \dot{\gamma}_x(0) \in T_{f(x)} N$$

identify $\dot{\gamma}_x(0) \equiv \dot{\gamma}(0)(x) \Rightarrow \dot{\gamma}(0) : M \rightarrow TN$

$\dot{\gamma}(0)(x) \in T_{f(x)} N$ i.e. $\dot{\gamma}(0)$ is a vector field along f

$$\Rightarrow T_f C(M, N) = \{\xi_f \in C(M, TN) \mid \tau_N \circ \xi_f = f\}$$

this is the parameter space at the point $f \in C(M, N)$

identify $T_f C(M, N) \cong \Gamma(f^* \tau_N)$ sections of pullback bundle ; ∞ -dim. vector space

different choices of topologies on $\Gamma(f^*\tau_M)$:

- C^k - topology $k < \infty, \Rightarrow \Gamma^k$ Banach space
 $\Rightarrow C^k(M, N)$ Banach manifold
- C^∞ - topology, $\Rightarrow \Gamma^\infty$ Frechet space, no norm
 $\Rightarrow C^\infty(M, N)$ Frechet manifold
- H^s - Sobolev topology $s > \frac{1}{2}\dim M, \Rightarrow \Gamma^S$ Hilbert space
 $\Rightarrow C^S(M, N)$ Hilbert manifold

note: for $N = M, f = id : M \rightarrow M$ we have
 $T_{id}C(M, M) = \Gamma(\tau_M) =$ vector fields on M

local charts (coordinates) of $C(M, M)$

Riemannian metric on M , then $\exp : TM \rightarrow M$

$\exp(v_x) = \alpha_x(1)$, $v_x \in T_x M$

$\alpha_x =$ geodesic at $x, \dot{\alpha}_x(0) = v_x$

\exp is a local diffeomorphism at $0 \in T_x M$, define
 $Exp := (\tau_M, \exp) : \mathcal{O}(0) \subset TM \rightarrow \mathcal{U}(\Delta) \subset M \times M$
diffeomorphism from a nbhd $\mathcal{O}(0)$ of zero section
onto $\mathcal{U}(\Delta)$ nbhd of diagonal $\Delta \subset M \times M$

chart: $\mathcal{V}(id_M) = \{f \in C(M, M) | graph(f) \subset \mathcal{U}(\Delta)\}$

$\mathcal{W}(0) = \{\xi \in C(M, TM) | \xi(M) \subset \mathcal{O}(0)\}$

open in $C(M, TM) \Leftrightarrow M$ **compact**

$\Phi : \mathcal{V}(id_M) \rightarrow \mathcal{W}(0) : \Phi(f) = Exp^{-1} \circ (id_M, f)$

bijection $\Phi^{-1}(\xi) = pr_2 \circ Exp \circ \xi$

we have a chart about the identity id_M
 $\Phi : \mathcal{V}(id_M) \subset C(M, M) \rightarrow \mathcal{W}(0) \subset \Gamma(\tau_M)$
 same construction for any $f, g \in C(M, M)$
 show transition maps : $\Gamma^s(f^*\tau_M) \rightarrow \Gamma^s(g^*\tau_M)$ smooth
 from Ω -Lemma

note: $\Gamma^s(f^*\tau_M)$, $s > \frac{1}{2}dimM$, Hilbert space
 $\Gamma^k(f^*\tau_M)$, $k < \infty$, Banach space
 $\Gamma^\infty(f^*\tau_M)$ Frechet space: differentiability ?

Theorem: $C^s(M, M)$, $s > \frac{1}{2}dimM$ is a smooth Hilbert manifold and $Diff^s(M) \subset C^s(M, M)$ is an open submanifold.

group operations are continuous

$$\mu : Diff^s(M) \times Diff^s(M) \rightarrow Diff^s(M)$$

$$\nu : Diff^s(M) \rightarrow Diff^s(M)$$

Theorem: $Diff^s(M)$ is a topological group

Lie group structure of $Diff^s(M)$

1) right multiplication: for $g \in Diff^s(M)$

$$R_g : Diff^s(M) \rightarrow Diff^s(M) : R_g(f) = f \circ g$$

derivative: $T_f R_g : T_f Diff^s(M) \rightarrow T_{f \circ g} Diff^s(M) :$

$$T_f R_g(\xi_f) = \xi_f \circ g, \text{ hence } TR_g = R_g \text{ so}$$

R_g is C^∞ for any $g \in Diff^s(M)$

2) left multiplication: for $g \in Diff^s(M)$

$$L_g : Diff^s(M) \rightarrow Diff^s(M) : L_g(f) = g \circ f$$

derivative: $T_f L_g : T_f Diff^s(M) \rightarrow T_{g \circ f} Diff^s(M) :$

$$T_f L_g(\xi_f) = Tg \circ \xi_f, \text{ hence } TL_g = L_{Tg} \text{ so}$$

$L_g : Diff^{s+k}(M) \rightarrow Diff^s(M)$ is C^k for any $g \in Diff^{s+k}(M)$, (Ω -Lemma)

together we get: multiplication is C^k

$$\mu : Diff^{s+k}(M) \times Diff^s(M) \rightarrow Diff^s(M)$$

$k = 0 \Rightarrow$ multiplication is continuous

similar the inversion is C^k

$$\nu : Diff^{s+k}(M) \rightarrow Diff^s(M)$$

$k = 0 \Rightarrow$ inversion is continuous

Lie algebra of $Diff^s(M)$

recall for a finite dim. Lie group G , the Lie algebra structure on $\mathfrak{g} \cong T_e G$ (e =identity) is defined as follows: $\xi, \eta \in T_e G$ define left invariant vector fields $X_\xi(g) := T_e L_g(\xi)$, $X_\eta(g) := T_e L_g(\eta)$, $g \in G$ define $[\xi, \eta] := [X_\xi, X_\eta](e)$

for $G = Diff^s(M)$ we use **right** invariant vector fields $Y : Diff^s(M) \rightarrow TDiff^s(M) = C^s(M, TM)$ $X(f) \in T_f Diff^s(M)$, i.e $\tau_M \circ X(f) = f$ vector field along f , and $T_{id_M} Diff^s(M) = \mathcal{X}^s(M)$, H^s -vector fields on M

define $Y_\xi(f) := TR_f(\xi) = \xi \circ f$, C^1 vector field on $Diff^s(M)$ if $\xi \in \mathcal{X}^{s+1}(M)$

$\xi, \eta \in \mathcal{X}^{s+1}(M) \Rightarrow [\xi, \eta] = -[Y_\xi, Y_\eta](id_M) \in \mathcal{X}^s(M)$ is a C^0 right invariant vector field on $Diff^s(M)$, so the ordinary Lie bracket $[\xi, \eta] = \xi\eta - \eta\xi$ of vector fields on M is the bracket on the Lie algebra $\mathfrak{g} = T_{id_M} Diff^{s+1}(M)$ but \mathfrak{g} is **not** closed under the bracket! \Rightarrow ILH Lie algebra!

exponential map $\exp : \mathfrak{g} \rightarrow \mathcal{G}$

$$\exp : T_e \text{Diff}^s(M) = \mathcal{X}^s(M) \rightarrow \text{Diff}^s(M)$$

$\xi \in \mathcal{X}^s(M)$, flow $\phi_t \in \text{Diff}^s(M)$. Then the curve $\phi(t) = \phi_t$ is integral curve of right invariant vector field Y_ξ on $\text{Diff}^s(M)$, indeed

$\phi(0) = \phi_0 = id_M$, and for $x \in M$

$$\frac{d}{dt}\phi(t)(x) = \frac{d}{dt}\phi_t(x) = \xi(\phi_t(x)) = (\xi \circ \phi(t))(x) = Y_\xi(\phi(t))(x). \text{ i.e. } \dot{\phi}(t) = Y_\xi(\phi_t) \text{ and } \dot{\phi}(0) = \xi.$$

define:

$$\exp(\xi) := \phi_1$$

then $\exp : \mathcal{X}^s(M) \rightarrow \text{Diff}^s(M)$ is continuous, but **not** C^1 , and **not** locally onto.

moral: $\text{Diff}^s(M) \neq \exp \mathcal{X}^s(M)$

- the diffeomorphism group is **not** generated by exponentiating vector fields
- in every neighborhood of the id_M there are diffeomorphisms which are not part of the flow of any vector field

ILH - Lie groups

Definition:

A collection of groups $\{G^\infty, G^s | s \geq s_0\}$ is called an **ILH-Lie group** (Inverse Limit of Hilbert) if:

(i) each G^s is a Hilbert manifold of class $C^{k(s)}$ modeled on a Hilbert space E^s , $k(s) \rightarrow \infty$ as $s \rightarrow \infty$

(ii) each $s \geq s_0$: $E^{s+1} \hookrightarrow E^s$ linear, continuous, dense inclusions, and

$G^{s+1} \hookrightarrow G^s$ dense inclusions of class $C^{k(s)}$

(iii) each G^s is a topological group and

$G^\infty = \varprojlim_s G^s$ is a topological group with the

inverse limit topology

(iv) (U^s, φ^s) chart of $G^s \Rightarrow (U^s \cap G^t, \varphi^s|_{U^s \cap G^t})$ is chart for G^t for all $t \geq s$

(v) multiplication $\mu : G^\infty \times G^\infty \rightarrow G^\infty$ extends C^k to $\mu : G^{k+s} \times G^s \rightarrow G^s$ for all $s, k \leq k(s)$

(vi) inversion $\nu : G^\infty \rightarrow G^\infty$ extends C^k to $\nu : G^{s+k} \rightarrow G^s$, for all $s, k \leq k(s)$

(vii) right multiplication $R_g, g \in G^s$ extends $C^{k(s)}$ to $R_g : G^s \rightarrow G^s$

ILH Lie algebras

Definition:

A collection of vector spaces $\{\mathfrak{g}^\infty, \mathfrak{g}^s | s \geq s_0\}$ is called an **ILH-Lie algebra** if

(i) each \mathfrak{g}^s is a Hilbert space and for each $s \geq s_0$ there are linear, continuous, dense inclusions $\mathfrak{g}^{s+1} \hookrightarrow \mathfrak{g}^s$ and $\mathfrak{g}^\infty = \varprojlim_s \mathfrak{g}^s$ is a Frechet space with the inverse limit topology

(ii) there exist bilinear, continuous, antisymmetric maps $[\cdot, \cdot] : \mathfrak{g}^{s+2} \times \mathfrak{g}^{t+2} \rightarrow \mathfrak{g}^{\min(s,t)}$, for all $s, t \geq s_0$, which satisfy the Jacobi identity on $\mathfrak{g}^{\min(s,t,r)}$ for elements in $\mathfrak{g}^{s+4} \times \mathfrak{g}^{t+4} \times \mathfrak{g}^{r+4}$.

Theorem: Let $\{G^\infty, G^s | s \geq s_0\}$ be an ILH-Lie group and let $\mathfrak{g}^s := T_e G^s$ and $\mathfrak{g}^\infty = \varprojlim_s \mathfrak{g}^s$.

Then $\{\mathfrak{g}^\infty, \mathfrak{g}^s | s \geq s_0\}$ is the ILH-Lie algebra of the ILH Lie group $\{G^\infty, G^s | s \geq s_0\}$.

Example : Diffeomorphism groups

$$\{Diff^\infty(M), Diff^s(M) | s > \frac{1}{2} \dim M\}$$

is an ILH Lie group with ILH-Lie algebra

$$\{\mathcal{X}^\infty(M), \mathcal{X}^s(M) | s > \frac{1}{2} \dim M\}$$

$k(s) = \infty$ for all s

Remarks:

1) Nash-Moser implicit function theorem works for ILH !

2) C^k -diffeomorphism groups, $0 \leq k < \infty$

$$\{Diff^\infty(M), Diff^k(M) | k \in \mathbf{N}\}$$

is an ILB Lie group (**I**nverse **L**imit of **B**anach) with ILB-Lie algebra (similar definitions as ILH)

$$\{\mathcal{X}^\infty(M), \mathcal{X}^k(M) | k \in \mathbf{N}\}$$

3) $Diff^\infty(M)$ considered as Frechet manifold and Frechet Lie group. Need differential calculus in Frechet spaces ! Not unique concept.

NO implicit (inverse) function theorem!

Gauge Groups

Let $\pi : P \rightarrow M$ be a principal G bundle
 G finite dimensional Lie group (structure group)
acting on P , $p \in P$, $g \in G$, $p \cdot g \in P$.

The **Gauge group** \mathcal{G} is the group of gauge transformations, i.e.

$$\mathcal{G} = \{\phi \in \text{Diff}^\infty(P) \mid \phi(p \cdot g) = \phi(p) \cdot g, \pi(\phi(p)) = \pi(p)\}$$

\mathcal{G} is canonically isomorphic to

$$\text{Gau}(P) = \{\tau \in C^\infty(P, G) \mid \tau(p \cdot g) = g^{-1} \tau(p)\}$$

Lie subgroup of loop group $C^\infty(P, G)$.

$\text{Gau}(P)$ isomorphic to $C^\infty(\text{Ad } P)$, sections of associated vector bundle $\text{Ad}(P) = P \times_G G$.

Complete in H^s Sobolev topology, set $\mathcal{G}^s = H^s(\text{Ad}P)$.

Gauge group \mathcal{G}^s is a smooth Hilbert Lie group, with smooth group operations.

Lie algebra \mathfrak{g} of gauge group:

a) \mathfrak{g} Lie algebra of \mathcal{G}

G -invariant, vertical vector fields X on P , i.e.

$$\mathfrak{g} = \{X \in \mathbf{X}^\infty(P) \mid R_a^* = X, X(p) \in \mathfrak{g}, a \in \mathfrak{g}\}$$

Commutator bracket

$$[X_1, X_2] = X_1X_2 - X_2X_1 \in \mathfrak{g}$$

b) $\mathfrak{gau}(P)$ Lie algebra of $\mathit{Gau}(P)$

Ad-invariant \mathfrak{g} -valued functions on P , i.e.

$$\mathfrak{gau}(P) = \{\xi \in C^\infty(P, \mathfrak{g}) \mid \xi(p \cdot a) = \text{Ad}_{a^{-1}}\xi(p)\}$$

Pointwise bracket $[\xi_1, \xi_2](p) = [\xi_1(p), \xi_2(p)]$

c) $C^\infty(\mathit{ad}(P))$ Lie algebra of $C^\infty(\mathit{Ad} P)$

sections of associated vector bundle

$\mathit{ad}(P) \equiv (P \times_G \mathfrak{g}) \rightarrow M$, pointwise bracket.

Proposition : The Lie algebras \mathfrak{g} , $gau(P)$ and $C^\infty(adP)$ are canonically isomorphic.

Proof: 1) $s \in C^\infty(adP)$ define $\xi \in gau(P)$.

$$\xi : P \rightarrow \mathfrak{g}, \xi(p \cdot a) = Ad_{a^{-1}}\xi(p)$$

$$\xi \in gau(P), \text{ define } s \in C^\infty(adP) \quad s(\pi(p)) = [p, \xi(p)].$$

2) $\xi \in gau(P)$ define $Z_\xi \in \mathfrak{g}$

$$Z_\xi(p) = \left. \frac{d}{dt} \right|_{t=0} R(p, \exp t\xi(p)), \quad (= \xi(p)^*(p)),$$

i.e. Z_ξ fundamental vector field on P , generated by $\xi \in \mathfrak{g}$.

Z_ξ is invariant iff $\xi(p \cdot a) = Ad_{a^{-1}}\xi(p)$.

Topologize \mathfrak{g} : complete $C^\infty(ad P)$ in H^s -Sobolev norm . If $s > \frac{1}{2}dimM$ then

$$\mathfrak{g}^s \simeq H^s(ad P) \simeq gau^s(P)$$

is a Hilbert Lie algebra.

Exponential map

$$Exp : gau^s(P) \rightarrow Gau^s(P)$$

$Exp : gau^s(P) \rightarrow Gau^s(P) : (Exp \xi)(p) = \exp(\xi(p))$
 $\xi \in gau^s(P)$ local diffeomorphism .

$Exp : \mathfrak{g}^s \rightarrow \mathcal{G}^s : (Exp \xi)(p) = p \cdot \exp(\xi_p)$.

Proposition: For $s > \frac{1}{2}dimM$

$$\mathcal{G}^s \simeq Gau^s(P) \simeq H^s(Ad P)$$

is a smooth Hilbert Lie group with Lie algebra

$$\mathfrak{g}^s \simeq gau^s(P) \simeq H^s(ad P)$$

and exponential map

$$EXP : \mathfrak{g}^s \rightarrow \mathcal{G}^s : (Exp \xi)(p) = p \cdot \exp(\xi(p))$$