

LECTURE 3

SUBGROUPS OF DIFFEOMORPHISM GROUPS and APPLICATIONS

SUBGROUPS OF DIFFEOMORPHISM GROUPS and APPLICATIONS

1. Volume preserving diffeomorphisms and fluid dynamics

M compact manifold, $\dim M = n$ and μ volume form on M i.e. μ n -forms, nondegenerate ($\mu(x) \neq 0 \forall x \in M$)

$$Diff_{\mu}^s(M) = \{f \in Diff^s(M) \mid f^*\mu = \mu\}$$

volume preserving H^s diffeomorphisms on M

$Diff_{\mu}^s(M) \subset Diff^s(M)$ subgroup ($(f \circ g)^* = f^* \circ g^*$)
is it a **LIE SUBGROUP** ?

look at Lie algebra i.e. tangent space $T_e Diff_{\mu}^s(M)$

let $\xi \in \mathcal{X}^s(M)$ with flow $\varphi_t \in Diff_{\mu}^s(M)$

since $\varphi_t^*\mu = \mu \Rightarrow 0 = \frac{d}{dt}|_{t=0} \phi_t^* = L_{\xi}\mu$ Lie derivative of μ along ξ . Divergence $L_{\xi}\mu = (div \xi)\mu$, so

$$T_e Diff_{\mu}^s(M) = \mathcal{X}_{\mu}^s(M) = \{\xi \in \mathcal{X}^s(M) \mid div \xi = 0\}$$

divergence free (incompressible) vector fields on M

$L_{[\xi, \eta]} = [L_{\xi}, L_{\eta}] \Rightarrow \mathcal{X}_{\mu}^s(M) \subset \mathcal{X}^s(M)$ sub Lie algebra

How do we show $Diff_\mu^s(M)$ is a Lie subgroup of $Diff^s(M)$?

Recall in classical (finite dim.) case we have:

- For any Lie algebra there is a Lie group
- False in ∞ dimensions !
- Any closed subgroup of a Lie group is a Lie subgroup. False in ∞ dimension !

Theorem: (Ebin-Marsden, 1970)

$Diff_\mu^s(M)$ is a closed smooth submanifold of $Diff^s(M)$

proof: (idea) 1) Hodge decomposition theorem
 $\Rightarrow [\mu]_s := \mu + d(H^{s+1}(\Omega^{n-1}(M)))$ closed subspace
in $H^s(\Omega^n(M)) = n$ - forms of class H^s

2) consider $F_\mu : Diff^s(M) \rightarrow [\mu]_s : F_\mu(f) = f^*\mu$
show F_μ is a smooth submersion

3) $T_\varphi F_\mu(\xi_\varphi) = \varphi^*(L_\xi\mu)$, $\xi_\varphi \in T_\varphi Diff^s(M)$

$\xi = \xi_\varphi \circ \varphi^{-1}$ hence F_μ is C^∞

for $\varphi = e : T_e F_\mu(\xi) = L_\xi\mu = di_\xi\mu + i_\xi d\mu$ but $d\mu = 0$
and μ nondegenerate $\Rightarrow \xi \in \mathcal{X}_\mu \mapsto i_\xi\mu \in H^s(\Omega^{n-1})$
isomorphism $\Rightarrow T_e F_\mu$ onto $\Rightarrow F_\mu$ submersion and

4) $F_\mu^{-1}(\mu) = Diff_\mu^s(M)$ is a closed submanifold of $Diff^s(M)$.

Theorem: $\{Diff_\mu^\infty(M), Diff_\mu^s(M) \mid s > \frac{1}{2}dim M\}$
is an ILH Lie group with

$\{\mathcal{X}_\mu^\infty(M), \mathcal{X}_\mu^s(M) , s > \frac{1}{2}dim M\}$ its ILH Lie algebra

Application: Fluid dynamics

$Diff_\mu^s(M)$ is the configuration space of incompressible, homogeneous, ideal fluids.

Euler equations:

$$\frac{\partial v}{\partial t} + \nabla_v v = -\nabla p, \quad \operatorname{div} v = 0$$

v = velocity field of fluid, vector field on M with flow η_t incompressible $\Rightarrow \eta_t \in Diff_\mu^s(M)$

$Diff_\mu^s(M)$ admits a Riemannian metric

$$(U, V)_\eta = \int_M \langle U(m), V(m) \rangle \eta(m) \mu(m)$$

$U, V \in T_\eta Diff_\mu^s(M)$, $\langle \cdot, \cdot \rangle$ metric on M

Consider geodesics V_t on $Diff_\mu^s(M)$, let $\eta_t = \tau_M \circ V_t$ i.e. $\dot{\eta}_t = V_t$ and set

$$v_t := V_t \circ \eta_t^{-1}$$

Theorem: (Ebin-Marsden 1970)

- v satisfies Euler equations $\Leftrightarrow \eta_t$ is a geodesic on $Diff_\mu^s(M)$
- existence of C^∞ geodesics on $Diff_\mu^s(M)$, small t
- Euler equations has unique solution for small t , depending C^∞ on initial condition v_0 .

Eichhorn-Schmid 2001, same for topological Euler equation on open manifolds

2. Canonical transformations (symplectomorphisms) and plasma physics

M compact manifold, $\dim M = 2n$, ω symplectic structure M i.e. nondegenerate closed ($d\omega = 0$) 2-form, consider

$$Diff_{\omega}^s(M) = \{f \in Diff^s(M) \mid f^*\omega = \omega\}$$

symplectomorphism group (canonical transform.) similar arguments as for volume preserving diffeomorphisms lead to

Theorem: $\{Diff_{\omega}^{\infty}(M), Diff_{\omega}^s(M) \mid s > n\}$ is an ILH-Lie subgroup of the ILH-Lie group $\{Diff^{\infty}(M), Diff^s(M) \mid s > n\}$, with ILH-Lie algebra $\{\mathcal{X}_{\omega}^{\infty}(M), \mathcal{X}_{\omega}^s(M) \mid s > n\}$

$$\mathcal{X}_{\omega}^s(M) = \{\xi \in \mathcal{X}^s(M) \mid L_{\xi}\omega = 0\}$$

locally Hamiltonian H^s - vector fields.

Application: Plasma physics (charges fluids)

let $f(x, v, t)$ be the density function

$E(x, t), B(x, t)$ the electric and magnetic fields

$q =$ charge, $m =$ mass :

Maxwell-Vlasov equations

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{q}{m}(E + v \times B) \frac{\partial f}{\partial v} = 0 \quad (\text{Boltzman})$$

$$\frac{\partial B}{\partial t} = -\text{curl } E$$

$$\frac{\partial E}{\partial t} = \text{curl } B - J_f, \text{ current density } J_f = q \int v f(x, v, t) dv$$

$$\text{div } E = \rho_f, \text{ charge density } \rho_f = q \int f(x, v, t) dv$$

$$\text{div } B = 0$$

Theorem : The Maxwell-Vlasov equations are an infinite dimensional Hamiltonian system, i.e. they can be written in the form

$$\dot{F} = \{F, H\}$$

for a certain non-canonical Poisson bracket $\{ , \}$ and some Hamiltonian H .

Poisson-Vlasov system: Limit case $B = 0$
 Maxwell-Vlasov system reduces to

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial v} - \frac{q}{m} \frac{\partial \phi_f}{\partial x} \cdot \frac{\partial f}{\partial x} = 0$$

Poisson-Vlasov equation

scalar potential ϕ_f given by $\Delta \phi_f = -\rho_f$

let $f(x, v, t) = \eta_t^* f(x, v, t_0) \Rightarrow \eta_t \in \text{Diff}_\omega(\mathbf{R}^6)$

identify Hamiltonian vector field $X_h(x, v)$ on \mathbf{R}^6
 with Hamiltonian $h(x, v) : \mathbf{R}^6 \rightarrow \mathbf{R}$, i.e. we have
 Lie algebra isomorphism

$$\mathfrak{g} = \mathcal{X}_\omega^\infty(\mathbf{R}^6) \cong C^\infty(\mathbf{R}^6), \quad [X_h, X_g] = X_{\{h, g\}}$$

we identify dual of Lie algebra \mathfrak{g}^* via L^2 -pairing

$$\langle h, f \rangle = \int h(x, v) f(x, v) dx dv \text{ with itself } \mathfrak{g}$$

now plasma density $f(x, v) \in \mathfrak{g}^* \cong C^\infty \mathbf{R}^6$

On \mathfrak{g}^* we have the Lie Poisson bracket:

$$F, G : \mathfrak{g}^* \rightarrow \mathbf{R}, \quad \{F, G\} = \langle \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}] \rangle, \quad \mu \in \mathfrak{g}^*$$

Poisson-Vlasov equation is in Lie-Poisson form on
 $\mathfrak{g}^* = C^\infty(\mathbf{R}^6)$ i.e. $\dot{F} = \{F, H\}$ with energy

$$H(f) = \frac{1}{2} \int mv^2 f(x, v, t) dx dv + \frac{1}{2} \int \phi_f \rho_f dx$$

and $\{, \}$ the Lie-Poisson bracket on $\mathfrak{g}^* = C^\infty(\mathbf{R}^6)$
 given by

$$\{F, G\}(f) = \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx dv.$$

Maxwell's equations:

$$\dot{E} = \text{curl } B, \quad \dot{B} = -\text{curl } E$$

are canonical Hamilton's equations on $T^*\mathcal{A}$ where the configuration space $\mathcal{A} = \{A : \mathbf{R}^3 \rightarrow \mathbf{R}^3\}$ vector potentials, $B = \text{curl } A$, so $(A, E) \in T^*\mathcal{A}$ and $H(E, B) = \frac{1}{2} \int (|E|^2 + |B|^2) dx$ is the total energy

Invariance of Maxwell's equations under gauge transformations $A \mapsto A + \nabla\phi$, $\phi \in C^\infty(\mathbf{R}^3)$ reduction procedure leads to

$$\text{div } E = \rho, \quad \text{div } B = 0$$

Coupled system: Maxwell-Vlasov equations

Same symmetry group $C^\infty(\mathbf{R}^6)$ acts on coupled phase space $C^\infty(\mathbf{R}^6) \times T^*\mathcal{A}$. The induced Poisson structure on the reduced phase space gives the Maxwell-Vlasov equations as Hamiltonian system

$$\dot{F} = \{F, H\}$$

with energy (Hamiltonian) $H(f, E, B) = \frac{1}{2} \int m v^2 f(x, v) dx dv + \frac{1}{2} \int |B(x)|^2 dx + \frac{1}{2} \int |E(x)|^2 dx$

and Poisson bracket:

For $F(f, E, B)$, $G(f, E, B)$

$$\begin{aligned}
\{F, G\}(f, E, B) &= \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} dx dv \\
&+ \int \left(\frac{\delta F}{\delta E} \cdot \text{curl} \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} \cdot \text{curl} \frac{\delta F}{\delta B} \right) dx \\
&+ \int \left(\frac{\delta F}{\delta E} \cdot \frac{\partial f}{\partial v} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta E} \cdot \frac{\partial f}{\partial v} \frac{\delta F}{\delta f} \right) dx dv \\
&+ \int f B \cdot \left(\frac{\partial}{\partial v} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial v} \frac{\delta G}{\delta f} \right) dx dv.
\end{aligned}$$

with this Poisson bracket the Maxwell-Vlasov equations are an infinite dimensional Hamiltonian system

$$\dot{F} = \{F, H\}$$

3) Contact transformations

M compact manifold, $\dim M = n$ and T^*M its cotangent bundle with canonical symplectic structure $\omega = d\theta = \sum dp_i \wedge dq_i$ (locally)

$\theta = \sum p_i \wedge dq_i$ canonical 1-form

consider diffeomorphisms

$$\varphi : T^*M \rightarrow T^*M, \quad \varphi^*\theta = \theta$$

$\Rightarrow \varphi = T^*\eta$ some $\eta \in \text{Diff}^s(M)$ extended point transformation. Delete the zero section in T^*M i.e. consider $T^*M - 0 = \dot{T}^*M$, then $\varphi^*\theta = \theta \Leftrightarrow \varphi^*\omega = \omega$ and $\varphi(\tau\alpha_x) = \tau\varphi(\alpha_x)$ all $\tau > 0, \alpha_x \in T_x^*M$ i.e. φ is symplectic and homogeneous of degree one. Consider

$$\text{Diff}_\theta^s(\dot{T}^*M) = \{\varphi \in \text{Diff}^s(\dot{T}^*M) \mid \varphi^*\theta = \theta\}$$

group of H^s contact transformations on \dot{T}^*M
note \dot{T}^*M is **not** compact

Theorem: (Ratiu-Schmid, 81)

$$\{\text{Diff}_\theta^\infty(\dot{T}^*M), \text{Diff}_\theta^s(\dot{T}^*M) \mid s > \dim M + 1\}$$

is an ILH Lie group.

sketch of proof: Main problem : T^*M is **not** compact, so previous method of describing $Diff_\theta^s(\dot{T}^*M)$ as a Lie subgroup of $Diff^s(\dot{T}^*M)$ fail.

Main idea: make T^*M compact i.e. pass to the cosphere bundle of M and show that $Diff_\theta^s(\dot{T}^*M)$ is isomorphic to the group of contact transformations (quantomorphisms) on the cosphere bundle.

Cosphere bundle $S(T^*M)$: \mathbf{R}_+ acts on \dot{T}^*M by $\alpha_x \mapsto \tau\alpha_x$. Orbit space $S(T^*M) := (\dot{T}^*M)/\mathbf{R}_+$ is **compact** smooth manifold if M compact

$\dim S(T^*M) = 2n - 1$. No canonical contact structure on $S(T^*M)$ but whole family given by

$\sigma : S(T^*M) \rightarrow \dot{T}^*M$ global section : $\theta_\sigma := \sigma^*\theta$

θ_σ is exact contact 1-form on $S(T^*M)$, i.e.

$\theta_\sigma \wedge (\theta_\sigma)^{n-1}$ is a volume form on $S(T^*M)$.

note: $\pi^*\theta_\sigma \neq \theta$, $\pi : \dot{T}^*M \rightarrow S(T^*M)$ but σ uniquely determined by function $f_\sigma : \dot{T}^*M \rightarrow \mathbf{R}$ defined by $\sigma(\pi(\alpha_x)) = f_\sigma(\alpha_x)\alpha_x$. Then

$$\pi^*\theta_\sigma = f_\sigma \cdot \theta$$

Contact transformations: Let $\eta \in Diff_{\theta}^s(\dot{T}^*M)$ and $\sigma : S(T^*M) \rightarrow \dot{T}^*M$ section.

$\eta^*\theta = \theta \Rightarrow \eta$ homogeneous degree 1 \Rightarrow exist unique $\varphi : S(T^*M) \rightarrow S(T^*M)$ defined by $\varphi \circ \pi = \pi \circ \eta$
 φ is contact transformation for θ_{σ} i.e. $\varphi^*\theta_{\sigma} = h\theta_{\sigma}$
 $h : S(T^*M) \rightarrow \mathbf{R}_+$

bijection $\eta \leftrightarrow (\varphi, h)$

composition \leftrightarrow semidirectproduct

$\eta_1 \leftrightarrow (\varphi_1, h_1)$, $\eta_2 \leftrightarrow (\varphi_2, h_2)$ then $\eta_1 \circ \eta_2 \leftrightarrow (\varphi_1, h_1) \cdot (\varphi_2, h_2) = (\varphi_1 \circ \varphi_2, h_2 \cdot (h_1 \circ \varphi_2))$ semidirect product

Theorem: 1) The group of contact transformations $Con_{\sigma}^s(S(T^*M)) =$

$\{(\varphi, h) \in Diff^s S(T^*M) \ltimes C^s(S(T^*M), \dot{\mathbf{R}}) \mid \varphi^*\theta_{\sigma} = h\theta_{\sigma}\}$

is a closed H^s Lie subgroup of the semidirect product $Diff^s S(T^*M) \ltimes C^s(S(T^*M), \dot{\mathbf{R}})$

2) $Diff_{\theta}^s(\dot{T}^*M)$ is isomorphic (as group) to $Con_{\sigma}^s(S(T^*M))$

Corresponding Lie algebras:

The Lie algebra of $Diff^s S(T^*M) \ltimes C^s(S(T^*M), \mathbf{R})$ is the semidirect product $\mathcal{X}^s(S(T^*M)) \ltimes C^s(S(T^*M), \mathbf{R})$ with bracket $[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f))$

The Lie algebra of $Con_\sigma^s(S(T^*M))$ is

$$con_\sigma^s(S(T^*M)) = \{(Y, g) \in \mathcal{X}^s(S(T^*M)) \ltimes C^s(S(T^*M), \mathbf{R}) \mid L_Y \theta_\sigma = g \theta_\sigma\}$$

On the other hand, the Lie algebra of $Diff_\theta^s(\dot{T}^*M)$ is $\mathcal{X}_\theta^s(S(T^*M)) = \{Y \in \mathcal{X}^s(S(T^*M)) \mid L_Y \theta = 0\}$

Now $L_Y \theta = 0 \Leftrightarrow Y$ is globally Hamiltonian vector field, homogeneous degree 0, i.e. $Y = Y_H$ and H homogeneous degree 1. $H = \theta(Y)$ unique.

Let $\mathcal{H}^s(\dot{T}^*M) = \{H \in C^s(T^*M, \mathbf{R}) \mid H \text{ homog. deg } 1\}$
Lie algebra with canonical Poisson bracket isomorphic to Lie algebra $\mathcal{X}_\theta^s(\dot{T}^*M)$ with commutator bracket $[X, Y] = XY - YX$.

Theorem: $Diff_\theta^s(\dot{T}^*M)$ and $Con_\sigma^s(S(T^*M))$ are isomorphic ILH-Lie groups with isomorphic ILH-Lie algebras $\mathcal{X}_\theta^s(\dot{T}^*M) \cong con_\sigma^s(S(T^*M)) \cong \mathcal{H}^s(\dot{T}^*M)$

Remark: $Diff_\theta^s(\dot{T}^*M)$ Fourier integral operators and $\mathcal{H}^s(\dot{T}^*M)$ symbols of pseudodifferential operators \Rightarrow quantization !

4) Globally Hamiltonian vector fields

Let (M, ω) be a compact symplectic manifold, $\mathcal{H}^\infty(M)$ the space of globally Hamiltonian vector fields on M . i.e. $X \in \mathcal{H}^\infty(M)$ iff $X = X_H$, $H : M \rightarrow \mathbf{R}$
 $\omega(X_H, Y) = dH \cdot Y \Leftrightarrow i_{X_H}\omega = dH$

$$X_{\{F, H\}} = [X_H, X_F]$$

hence $\mathcal{H}^\infty(M)$ is a Lie subalgebra of $\mathcal{X}^\infty(M)$.

Question: Is there a corresponding Lie group ?

Fact: (Calabi, Arnold) $\mathcal{H}^\infty(M) \subset \mathcal{X}_\omega^\infty(M)$ locally Hamiltonian vector fields and the commutator algebra $[\mathcal{X}_\omega^\infty(M), \mathcal{X}_\omega^\infty(M)] = \mathcal{H}^\infty(M)$

Let $Diff_\omega^\infty(M)_0 =$ identity component of $Diff_\omega^\infty(M)$

Theorem: (Ratiu, Schmid 81) The commutator subgroup $[Diff_\omega^\infty(M)_0, Diff_\omega^\infty(M)_0]$ is a simple, closed ILH-Lie subgroup of $Diff_\omega^\infty(M)_0$ with ILH Lie algebra $\mathcal{H}^\infty(M) = [\mathcal{X}_\omega^\infty(M), \mathcal{X}_\omega^\infty(M)]$.

(Recall: $[\mathfrak{g}, \mathfrak{h}]$ generated by $[X, Y]$, $X \in \mathfrak{g}, Y \in \mathfrak{h}$,
 $[G, H]$ generated by $ghg^{-1}h^{-1}$, $g \in G, h \in H$)

Proof:(idea, Banyaga, Calabi) consider the map

$$S : Diff_{\omega}^{s+1}(M)_0 \rightarrow H^1(M, \mathbf{R}) , S(h) = [A(h_t)]$$

where $h_t =$ symplectic homotopy from h to id_M with locally Hamiltonian vector field $X_t = \frac{dh_t}{dt} \circ h_t^{-1}$. Define $A(h_t) = \int_0^1 i(X_t)\omega dt$ closed H^{s+1} -one form defining a cocycle in $H^1(M, \mathbf{R})$.

Show the following:

- S is a group homomorphism and $Ker S$ is perfect.
 $H^1(M, \mathbf{R})$ abelian $\Rightarrow Ker S = [Diff_{\omega}^{\infty}(M)_0, Diff_{\omega}^{\infty}(M)_0]$
- S is a C^{∞} submersion $\Rightarrow Ker S$ is a closed H^{s+1} -Lie subgroup of $Diff_{\omega}^{s+1}(M)_0$

5) The group of quantomorphisms

Let (M, θ) be a compact exact contact manifold, i.e. M smooth $(2n + 1)$ dim., $\theta = 1$ -form such that $\theta \wedge (d\theta)^n = \text{volume element on } M$. Consider

$$Diff_{\theta}^s(M) = \{\eta \in Diff^s(M) \mid \eta^*\theta = \theta\}$$

called the quantomorphism group of M : why ?

characteristic integrable line bundle of $d\theta$

$$R_{d\theta} = \{v \in TM \mid i_v d\theta = 0\}$$

Reeb vector field E of $R_{d\theta}$ def. by $i_E \theta = 1, i_E d\theta = 0$ locally $(x^1, \dots, x^n, y^1, \dots, y^n, t), \theta = \sum_{i=1}^n y^i dx^i + dt$
 $E = \frac{\partial}{\partial t}$

characteristic integrable $2n - \text{dim}$ bundle of θ

$$R_{\theta} = \{v \in TM \mid \theta(v) = 0\}$$

$TM = R_{d\theta} \oplus R_{\theta}$ hence $\mathcal{X}^s(M) = \mathcal{X}^s(R_{d\theta}) \oplus \mathcal{X}^s(R_{\theta})$

The leaves of foliation \mathcal{F} defined by line bundle $R_{d\theta}$ (circle action) are integral curves of E . Quotient $N = M/\mathcal{F}$ is smooth symplectic manifold with $\pi : M \rightarrow N, \pi^*\omega = \theta$, so $\theta =$ connection 1-form of this principal circle bundle with horizontal bundle $= R_{\theta}$ and $\omega =$ curvature 2-form.

Thus M is the **quantizing manifold** of N .

let $\mathcal{K}^s(N) = \{\varphi \in Diff_\omega^s(N) \mid H_{\varphi \circ \gamma} = H_\gamma = \text{horizontal transport along } \gamma\}$

We have the following reformulation of Kostant's prequantization theorem:

Theorem: The following sequence of groups is exact

$$0 \longrightarrow S^1 \xrightarrow{J} Diff_\theta^s(M) \xrightarrow{P} \mathcal{K}^s(N) \longrightarrow 0$$

the following sequence of Lie algebras is a exact

$$0 \longrightarrow \mathbf{R} \xrightarrow{j} \mathcal{X}_\theta^s(M) \xrightarrow{p} \mathcal{H}^s(N) \longrightarrow 0$$

Theorem: (Ratiu, Schmid 81) The quantomorphism group $Diff_\theta^\infty(M)$ is an ILH principal circle bundle over the ILH-Lie group $\mathcal{K}^\infty(N)$ with ILH Lie algebra $\mathcal{X}_\theta^s(M) = \{X \in \mathcal{X}^\infty(M) \mid L_X\theta = 0\}$ the infinitesimal quantomorphisms.

6) The group of gauge transformations and quantum field theory

Principal G -bundle $\pi : P \rightarrow M$, group \mathcal{G} of gauge transformations

$$\mathcal{G} = \{\phi \in \text{Diff}^\infty(P) \mid \phi(p \cdot g) = \phi(p) \cdot g, \pi\phi(p) = \pi(p)\} \\ \cong \{\tau \in C^\infty(P, G) \mid \tau(p \cdot g) = g^{-1}\tau(p)\}$$

Hilbert Lie group with smooth group operations.

Application: Quantum chromo dynamics (QCD) and Yang-Mills theory:

\mathcal{A} = space of connection 1-forms on P (vector potentials) . $A \in \mathcal{A} \Rightarrow D_A$ covariant differential curvature 2-form (field strength)

$$F_A = D_A A = dA + \frac{1}{2}[A, A]$$

locally $A = A_\mu dx^\mu$, $F = \frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu$

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} + [A_\mu, A_\nu]$$

Gauge group \mathcal{G} acts on \mathcal{A} via pull-back

$$\phi \in \mathcal{G}, A \in \mathcal{A}, \phi \cdot A = \phi^* A \in \mathcal{A}$$

locally

$$\tau \cdot A = \tau^{-1} d\tau + \tau^{-1} A \tau$$

hence $D_{\tau \cdot A} = \tau^{-1} D_A \tau$ and action of the field

$$\tau \cdot F_A := F_{\tau \cdot A} = \tau^{-1} F_A \tau$$

Action functional (Yang-Mills functional)

$$S(A) = \|F_A\|^2$$

locally: $\|F_A\|^2 = \frac{1}{2} \int_M \text{tr}(F_{\mu\nu} F^{\mu\nu})$

Action is gauge invariant $S(\phi \cdot A) = S(A)$, $\phi \in \mathcal{G}$

so Yang-Mills functional is defined on orbit space

$$\mathcal{M} = \mathcal{A}/\mathcal{G}.$$

\mathcal{M} in general not a manifold since action of \mathcal{G} on \mathcal{A} not free. Restricts to irreducible connections then \mathcal{M} is smooth infinite dimensional manifold and $\mathcal{A} \rightarrow \mathcal{M}$ infinite dimensional principal fiber bundle with structure group \mathcal{G} .

For self-dual connections $F_A = *F_A$, (instantons) on compact 4-manifold the moduli space

$$\mathcal{M} = \{A \in \mathcal{A} | A \text{ self-dual}\} / \mathbf{G}$$

is a smooth finite dimensional manifold

Self-dual connections absolutely minimize Yang-Mills action integral

$$\text{YM}(A) = \int_{\Omega} \|F_A\|^2, \quad \Omega \subset M \text{ compact}$$

Feynman integral: Quantize the action get probability amplitude

$$\int_{\mathcal{A}/\mathbf{G}} e^{-S(A)} f(A) \mathbf{D}(A)$$

for gauge invariant functionals $f(A)$