LECTURE 4

Lie GROUPS of FOURIER INTEGRAL OPERATORS

Lie groups of pseudodifferental operators and Fourier integral operators

Consider differential operator P on $\Omega \subset \mathbf{R}^n$ of order m with smooth coefficients a_α

$$Pu(x) = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha u(x) \ , \quad u \in C^\infty(\Omega)$$

the **symbol** of P is the polynomial

$$p(x,\xi) \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$$

Fourier transform $\hat{u}(\xi)$ of $u(x)$ we have $\widehat{D_x^\alpha} u(\xi) = \xi^\alpha \hat{u}(\xi)$ and $D_x^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi)\, d\xi$ so we write

$$\begin{aligned} Pu(x) &= (2\pi)^{-n} \int e^{ix\cdot\xi}\ p(x,\xi)\hat{u}(\xi)\ d\xi \\ &= (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi}\ p(x,\xi) u(y)\ dy d\xi \end{aligned}$$

a **pseudodifferential operator** P is of this form but with symbol $p(x,\xi)$ of more general class then polynomials. A smooth function $p(x,\xi)$ on $\Omega \times \mathbf{R}^n$ belongs to symbol class $S^m_{\rho,\delta}(\Omega)$, $0 \le \delta < \rho < 1$ if for any compact $K \subset \Omega$, any α, β exists const. $C_{\alpha\beta}(K) > 0$ such that for all $x \in K, \xi \in \mathbf{R}^n$

$$|D_x^\beta D_\xi^\alpha\ p(x,\xi)| \le C_{\alpha,\beta}(K)(1 + |\xi|)^{m - \rho|\alpha| + \delta|\beta|}$$

classical symbol has asymptotic expansion

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi)\ ,\ m = \text{order of } p$$

each $p_{m-j}(x,\xi) \in C^\infty(\Omega \times \dot{\mathbf{R}}^n)$ homogeneous of degree $m-j$, $\ p_{m-j}(x,\tau\xi) = \tau^{m-j} p_{m-j}(x,\xi), \tau > 0$
A classical ΨDO of order m is of the form

$$Pu(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi}\ p(x,\xi)u(y)\ dyd\xi$$

with $p(x,\xi)$ a classical symbol.
$p_m(x,\xi)$ = principal symbol of P
Denote ΨDO_m space of classical ΨDO of order m and $\Psi DO = \bigcup_m \Psi DO_m$ infinite dim. Lie algebra of all ΨDO with commutator bracket $P \in \Psi DO_m$ $Q \in \Psi DO_n \Rightarrow [P,Q] \in \Psi DO_{m+n+-1}$
note: ΨDO_1 is a Lie subalgebra of ΨDO
question: are there Lie groups corresponding to these Lie algebras ?
answer: YES ! FIO_0 the groups of invertible Fourier integral operators of order 0 for ΨDO_1 and FIO the groups of all invertible Fourier integral operators for ΨDO.
We discuss these Lie group structures now:
1. What is a Fourier integral operator ?

generating functions for canonical transformations:

Let $S : \Omega \times \mathbf{R}^n \to \mathbf{R}$ be smooth in a nbhd of $(x_0, \xi_0) \in \Omega \times \mathbf{R}^n$ such that $\frac{\partial^2 S(x,\xi)}{\partial x \partial \xi} \neq 0$. Then $\Phi(y, \xi) = (x, \eta)$ where $\eta = \frac{\partial S(x,\xi)}{\partial x}$, $y = \frac{\partial S(x,\xi)}{\partial \xi}$ defines canonical transformation

$$\Phi : (y_0, \xi_0) \in \dot{T}^* \mathbf{R}^n \to (x_0, \eta_0) \in \dot{T}^* \Omega \ , \quad \Phi^* \omega = \omega$$

S = generating function of Φ; every canonical transformation Φ has a locally generating function

Example: $S(x, \xi) = x \cdot \xi \Rightarrow \eta = \frac{\partial S}{\partial x} = \xi$, $y = \frac{\partial S}{\partial \xi} = x$ hence $\Phi = id$.

Fourier integral operators, FIO

Let $S(x, \xi)$ generating function and $a(x, \xi)$ classical symbol order m. Define an classical Fourier integral operator A of order m by

$$\begin{aligned} Au(x) \ &:= \int e^{iS(x,\xi)} a(x,\xi) \hat{u}(\xi) \ d\xi \\ &= (2\pi)^{-n} \int \int e^{i(S(x,\xi) - y \cdot \xi)} a(x,\xi) u(y) \ dy d\xi \end{aligned}$$

More generally , a Fourier integral operator A of order m is defined by

$$Au(x) = (2\pi)^{-n} \int\int e^{i\varphi(x,y,\xi)} a(x,\xi) u(y)\ dyd\xi$$

where $\varphi(x,y,\xi)$ is nondegenerate **phase function** (homogeneous +1) and the symbol $a(x,\xi) \in S^m_{\rho,\delta}$

notice: if $S(x,\xi) = x \cdot \xi$ or general $\varphi(x,y,\xi) = (x-y)\cdot\xi$ then the operator A is a ΨDO, we have

$$FIO \supset \Psi DO \supset DO$$

FIO are singular operators with nice properties

1) invariant under diffeomorphisms $\Rightarrow$ can be defined on manifolds M as bounded linear operators

$$A : C^\infty(M) \to C^\infty(M), \quad M \text{ compact!}$$

such that A is locally of the form above, moreover extend to distributions $A : \mathcal{E}'(M) \to \mathcal{D}'(M)$

$P \in \Psi DO_m$ extends $P : H^s_c(M) \to H^{s-m}_c(M)$, bounded

2) properties close to differential operators (DO)

$P \in DO \Leftrightarrow P$ is local, i.e. $supp\ Pu \subset supp\ u$

$P \in \Psi DO \Rightarrow P$ pseudolocal, $sing\ supp\ Pu \subset sing\ supp\ u$
P preserves wave front sets WF, $WF(Pu) \subset WF(u)$
where $WF(u) \subset \dot{T}^*M, \quad \tau_M^* WF(u) = sing\ supp\ u$

$A \in FIO$ moves the wave front set by canonical relation Λ : $\quad WF(Au) \subset \Lambda \circ WF(u) \quad$ where
$\Lambda \subset \dot{T}^*M \times \dot{T}^*M$ is a conic Lagrangian submanifold locally generated by phase function $\varphi(x, y, \xi)$

$$\Lambda = \{(x, y, d_{(x,y)}\varphi(x, y, \xi)) \mid d_\xi \varphi = 0\}$$

Remarks:
1) if $\Lambda = \Delta = ((x, \xi), (x, \xi))$ diagonal $\Rightarrow A \in \Psi DO$
2) $A \in FIO$ is determined by symbol $a(x, \xi)$ and canonical relation Λ
3) principal symbol is globally defined
$a_m(x, \xi) : \dot{T}^*M \to \mathbf{R}$

3) closed under multiplication, let $\Phi : \dot{T}^*M \to \dot{T}^*M$
$\Phi^*\omega = \omega$ locally generated by $S(x, \xi)$
denote $FIO_m(\Phi)$= space of FIO order m
associated to canonical relation $\Lambda = graph(\Phi)$

If $A_1 \in FIO_{m_1}(\Phi_1),\ A_2 \in FIO_{m_2}(\Phi_2)$ then
$A_1 \circ A_2 \in FIO_{m_1+m_2}(\Phi_1 \circ \Phi_2)$

if $A \in FIO_m(\Phi)$ and $A^{-1} \in FIO$ exists,
then $A^{-1} \in FIO_m(\Phi^{-1})$.

Note $\Phi = id : \dot{T}^*M \to \dot{T}^*M \Rightarrow FIO_m(id) = \Psi DO_m$

Notation: FIO_* , ΨDO_* , $(FIO_m)_*$, $(\Psi DO_m)_*$
invertible elements
FIO_* , ΨDO_* , $(FIO_0)_*$, $(\Psi DO_0)_*$ are groups

Example: Let $f : M \to M$ be a diffeomorphism. Then

$$f^*u(x) = (2\pi)^{-n} \int\int e^{i(f(x)-y)\cdot\xi} u(y)dy\ d\xi$$

defines a FIO $f^* : C^\infty(M) \to C^\infty(M)$ whose phase function generates the canonical cotangent lift
$T^*f : \dot{T}^*M \to \dot{T}^*M$

exact sequence

let $S(x, \xi)$ be generating function of Φ , $\Phi^* \omega = \omega$
S homogeneous $+1$ in ξ $\Rightarrow \Phi$ homog. $+1$ in ξ
hence $\Phi^* \theta = \theta$ canonical 1-from on $T^* M$
i.e. $\Phi \in Diff_\theta^\infty(\dot{T}^* M)$. Get surjective map

$$p : FIO \rightarrow Diff_\theta^\infty(\dot{T}^* M) \ , \ p(A) = \Phi$$

where $graph(\Phi) = \Lambda$ canonical relation of A
kernel of p is $p^{-1}(e) = \Psi DO_*$, $e = id_{\dot{T}^* M}$
ΨDO_* and FIO_* are groups under operator multiplication, graded by order (additive) and p is group homomorphism $p(A \circ B) = p(A) \circ p(B)$

exact sequence of groups

$$I \longrightarrow \Psi DO_* \overset{j}{\hookrightarrow} FIO_* \overset{p}{\longrightarrow} Diff_\theta^\infty(\dot{T}^* M) \longrightarrow e$$

we want exact sequence of **LIE GROUPS**

Notice: zero order operators are groups and form exact sequence

$$I \longrightarrow (\Psi DO_0)* \overset{j}{\hookrightarrow} (FIO_0)* \overset{p}{\longrightarrow} Diff_\theta^\infty(\dot{T}^* M) \longrightarrow e$$

we make this zero order operators into ILH Lie groups, then move structures by fixed elliptic operator T to any order m, e.g. $T = (1 + \Delta)^{m/2}$

what are parameter spaces ? Lie algebras ? of

$$I \longrightarrow (\Psi DO_0)* \overset{j}{\hookrightarrow} (FIO_0)* \overset{p}{\longrightarrow} Diff_\theta^\infty(\dot{T}^*M) \longrightarrow e$$

Lie algebras

$$I \longrightarrow \Psi DO_0 \overset{j}{\hookrightarrow} \Psi DO_1 \overset{\pi}{\longrightarrow} C_{+1}^\infty(\dot{T}^*M) \longrightarrow e$$

$\pi(P)=$ principal symbol (homog. +1)
$C_{+1}^\infty(\dot{T}^*M) \cong \mathcal{X}_\theta^\infty(\dot{T}^*M) = \{X | L_X\theta = 0\}$ globally Hamiltonian vector fields (Egorov's theorem)

Idea: we construct a principal fiber bundle with

- base space $= Diff_\theta^\infty(\dot{T}^*M)$
- total space $= (FIO_0)_*$
- fiber $= p^{-1}(\Phi) = (FIO_0(\Phi))_* \cong (\Psi DO_0)_*$
- structure group $=(\Psi DO_0)_*$

step1: $Diff_\theta^\infty(\dot{T}^*M) = \lim\limits_{\longleftarrow} Diff_\theta^s(\dot{T}^*M)$ ILH Lie group
step2: $(\Psi DO_0)_* = \lim\limits_{\longleftarrow}(\Psi DO_0^s)_*$ ILH Lie group
step3: piece 1 & 2 together via local section σ

$$\sigma : U \subset Diff_\theta^\infty(\dot{T}^*M) \to (FIO_0)_*$$

then $(FIO_0)_*$ locally: $p^{-1}(U) \simeq U \times (\Psi DO_0)_*$

$\Rightarrow$ chart at identity $I \in (FIO_0)_*$

step4: move this chart around by group structure of $Diff^s_\theta(\dot{T}^*M) \Rightarrow (FIO_0)_*$ topological group

step 5: chart transitions smooth $\Rightarrow (FIO_0)_*$ smooth manifold

step 6: multiplication "smooth" $\Rightarrow (FIO_0)_*$ Lie group

step 7: identify $(1-\Delta)^{m/2} : (FIO_0)_* \xrightarrow{\simeq} (FIO_m)_*$

Theorem: (M. Adams, T. Ratiu, R. Schmid, 1985) The group $FIO_*(M)$ of invertible Fourier integral operators on a **compact** manifold M is a graded ILH-Lie group with graded ILH-Lie algebra $\Psi DO(M)$ of pseudodifferential operators on M. $FIO_*(M)$ is and ∞-dim principal fiber bundle over the base manifold $Diff^s_\theta(\dot{T}^*M)$ of contact transformations of $\dot{T}^*M$ with gauge group $\Psi DO_*(M)$ of invertible pseudodifferential operators.

Step 1: $Diff_\theta^\infty(\dot{T}^*M)$ **as ILH Lie group**

Theorem: $Diff_\theta^\infty(\dot{T}^*M) = \lim_{\infty \leftarrow s} Diff_\theta^s(\dot{T}^*M)$ is an ILH Lie group where $Diff_\theta^s(\dot{T}^*M)$ is isomorphic to the semidirect product

$$Diff_\theta^s(ST^*M) = \{(\varphi, h) \in Diff^s(ST^*M) \ltimes C^s(ST^*M) \mid \varphi^*\theta_S = h\theta_S\}$$

with ILH Lie algebra

$\mathcal{X}_\theta^s(S(T^*M)) = \{Y \in \mathcal{X}^s(S(T^*M)) \mid L_Y\theta = 0\}$
isomorphic to
$C_{+1}^s(\dot{T}^*M) = \{H \in C^s(T^*M, \mathbf{R}) | H$ homog.deg1$\}$

Step 2: $(\Psi DO_0)_*$ **as ILH Lie group**
topology determined by the symbols $P \in \Psi DO_0$
$p(x,\xi) = \sum_{j=0}^{-\infty} p_j(x,\xi)$ infinitely many term
$\Rightarrow$ Frechet space. We cut the symbol at the term p_{-k} , fixed $k < \infty$. In terms of operators : quotient spaces $\Psi DO_{m,k} = \Psi DO_m / \Psi DO_{-k-1}$. Similar for FIO take $FIO_{m,k}(\eta) = FIO_m(\eta)/FIO_{-k-1}(\eta)$ and $FIO_{m,k} = \bigcup_\eta FIO_{m,k}(\eta)$, where
$FIO_m(\eta) = \{A \in FIO_m \mid p(A) = \eta \in Diff_\theta^\infty\}$

Composition well defined in $\Psi DO_{0,k}$ and $FIO_{0,k}$, denote by $(\Psi DO_{0,k})_*$ and $(FIO_{0,k)_*}$ the groups of invertible elements We still have the exact sequence of groups:

$$I \to (\Psi DO_{0,k})_* \stackrel{j}{\hookrightarrow} (FIO_{0,k})_* \stackrel{p}{\to} Diff_\theta^\infty(\dot{T}^*M) \to id$$

For $P \in \Psi DO_{m,k}$ with symbol $p(x,\xi) = p_m(x,\xi) + \cdots + p_{-k}(x,\xi)$ we define the norm by

$$\|P\|^2_{m+k,s} = \|\tilde{p}_m\|^2_{s+k+m} + \|\tilde{p}_{m-1}\|^2_{s+k+m-1} + \cdots + \|\tilde{p}_k\|^2_s$$

where $\tilde{p}_{m-j}$ is the restriction of p_{m-j} to ST^*M and $\|\tilde{p}_{m-j}\|^2_{s+k+m-j}$ is the $H^{s+k+m-j}$-Sobolev norm on ST^*M. Let $\Psi DO^s_{m,k}$ be the completion of $\Psi DO_{m,k}$ and $(\Psi DO^s_{0,k})_*$ the group of invertible elements in $\Psi DO^s_{0,k}$.

Theorem: (Adams,Ratiu,Schmid,1986) For each $s > n$ the group $(\Psi DO^s_{0,k})_*$ is a Hilbert Lie group with Lie algebra $\Psi DO^s_{0,k}$. That means $(\Psi DO^s_{0,k})_*$ is a smooth (C^∞) Hilbert manifold with smooth group operations. Moreover the inverse limit $(\Psi DO_{0,k})_* = \lim_{\infty \leftarrow s} (\Psi DO^s_{0,k})_*$ is an ILH Lie group .

At the end of the day we will take the limit $k \to \infty$!

Step 3: Local section: $\mathcal{U}$ nbhd of $id \in Diff_\theta^\infty(\dot{T}^*M)$

$$\sigma : \mathcal{U} \subset Diff_\theta^\infty(\dot{T}^*M) \to (FIO_0)_*$$

this gives $(FIO_0)_*$ local product structure

$$p^{-1}(\mathcal{U}) \cong \mathcal{U} \times (\Psi DO_0)_*$$

defined by $A \mapsto (p(A), A \circ \sigma(p(A))^{-1})$, and inverse $(\varphi, P) \mapsto P \cdot \sigma(\varphi)$.
We get chart at identity $I \in (FIO_0)_*$

problem: FIO are locally defined, need **global** writing of FIO i.e. global phase function for FIO close to I. This is done by constructing explicit chart about id of $Diff_\theta^s(\dot{T}^*M)$.

Theorem:(Adams-Ratiu-Schmid, 86)
Let $H \in C^S_{+1}(\dot{T}^*M)$ close to zero. define
$\varphi_H : \dot{T}^*M \times M \to \mathbf{R}$
$\varphi_H(\alpha_x, y) := \alpha_x \cdot (exp_x^{-1}(y)) + H(\alpha_x)$
Then there ex. $\Phi \in Diff_\theta^s(\dot{T}^*M)$ close to id such that φ_H is **global** phase function for $graph(\Phi)$
$H \leftrightarrow \Phi$ bijection $H(\alpha_x) = -\alpha_x \cdot exp_x^{-1}(\tau^*\Phi^{-1}(\alpha_x))$

define local section σ as follows: let $\eta \in Diff_{\theta}^{\infty}(\dot{T}^*M)$ close to the identity and define $\sigma(\eta)$ by

$$\sigma(\eta)u(x) :=$$
$$(2\pi)^{-n} \int\limits_{T_x^*M} \int\limits_{B_\delta(x)} \chi(x,y) e^{i\varphi_H(\alpha_x,y)} u(y) |\det \exp_x| dy d\xi$$

$\sigma(\eta)$ is a FIO with smooth phase function φ_H and amplitude $a = 1$. Moreover, $\sigma(\eta)$ is invertible modulo smoothing operators since η is invertible and its principal symbol is $a = 1$, hence $\sigma(\eta) \in (FIO_0)_*$. Furthermore, $p\sigma(\eta) = \eta$, hence σ is a local section of the exact sequence. We use this local section σ to give $(FIO_0)_*$ the local product structure

$$p^{-1}(U) \simeq U \times (\Psi DO_0)_* \ .$$

define topology around identity in $(FIO_{0,k})_*$ by the bijection $\Phi : p^{-1}(U^{2t}) \to U^{2t} \times (\Psi DO_{0,k}^{2(t-k)})_*$
$\Phi(A) = (p(A), A \circ \sigma(p(A))^{-1})$ and
$\Phi^{-1}(\eta, P) = P \circ \sigma(\eta)$, where $U^{2t} = U \cap Diff_{\theta}^{2t}$.

This defines a local chart at the identity $I \in (FIO_{0,k})_*$

Step 4: $(FIO_0)_*$ as topological group

To define the topology on $(FIO_{0,k})_*$ we move the open sets $p^{-1}(U^{2t})$ by right translations. Complete this topological space in the right-uniform structure and denote it by $(FIO^t_{0,k})_*$. For each $t > n/2$ we obtain $(FIO^t_{0,k})_*$ as a topological group and $(FIO_{0,k})_* = \bigcap_t (FIO^t_{0,k})_*$ with the inverse limit topology is a topological group as well.

To prove this, we have to show that the map $(A, B) \mapsto AB^{-1}$ is continuous for any $A, B \in (FIO^t_{0,k})_*$. This amounts to show that the following map in local coordinates is continuous:

$$(U^{2t} \times \psi^{2(t-k)}_{0,k}) \times (U^{2t} \times \Psi DO^{2(t-k)}_{0,k}) \to (U^{2t} \times \Psi DO^{2(t-k)}_{0,k})$$

$$((\eta_1, P_1), (\eta_2, P_2)) \mapsto (\eta_1 \circ \eta_2^{-1}, \quad P_1 \sigma(\eta_1) \sigma(\eta_2)^{-1} P_2^{-1} \sigma(\eta_1 \circ \eta_2^{-1})^{-1})$$

which involves a careful study of products of FIOs.

Step 5: $(FIO_0)_*$ as smooth manifold

Overlap conditions in local charts give conditions on σ to make $(FIO_0)_*$ into a smooth manifold. To prove that the transition maps between local charts are smooth we have to show that the following map is differentiable

$$(U^{2t} \cdot \alpha \cap U^{2t} \cdot \beta) \times (\Psi DO_{0,k}^{2(t-k)})_* \to (\Psi DO_{0,k}^{2(t-k)})_*$$

$$(\eta, P) \mapsto P\sigma(\eta \circ \alpha^{-1}) AB^{-1} \sigma(\eta \circ \beta^{-1})^{-1}$$

for any $A, B \in (FIO_{0,k}^t)_*$, where $\alpha = p(A)$, $\beta = p(B)$. The symbol calculus shows that this map is of class C^t , hence $(FIO_{0,k}^t)_*$ is a smooth manifold of class C^t.

Step 6: $(FIO_0)_*$ as ILH Lie group

We check smoothness of multiplication and inversion

$$\mu : (FIO_0)_* \times (FIO_0)_* \to (FIO_0)_* \ , \ \mu(A, B) = A \circ B$$

$$\nu : (FIO_0)_* \to (FIO_0)_* \quad , \quad \nu(A) = A^{-1}$$

To show that group multiplication in $(FIO^t_{0,k})_*$ is smooth we have to show that the following map is differentiable

$$(U^{2(t+s)} \cdot \alpha) \times (\Psi^{2(t+s+k)}_{0,k})_* \times (U^{2(t+s)} \cdot \beta) \times (\Psi^{2(t+s+k)}_{0,k})_*$$

$$\to (U^{2(t+s)} \cdot (r\alpha \cdot \beta) \times (\Psi^{2(t+s+k)}_{0,k})_*$$

$$((\eta_1, P_1), (\eta_2, P_2)) \mapsto$$

$$(\eta_1 \circ \eta_2, P_1 \sigma(\eta_1 \circ \alpha^{-1}) A P_2 \sigma(\eta_2 \beta^{-1}) A^{-1} \sigma(\eta_1 \eta_2 \beta^{-1} \alpha^{-1})^{-1}$$

for any $A \in (FIO^{t+s}_{0,k})_*, B \in (FIO^t_{0,k})_*$,
where $\alpha = p(A), \beta = p(B)$.

This makes $(FIO_0)_* = \lim_{\infty \leftarrow s} (FIO^s_0)_*$ into an ILH Lie group.

Step 7: FIO_* as Lie group

$(FIO_0)_*$ as a Lie group . To obtain a Lie group structure on all FIO_* we use the Laplace operator to identify $(1-\Delta)^{m/2} : (FIO_0)_* \xrightarrow{\sim} (FIO_m)_*$. Multiplication is smooth between the appropriate spaces.

The final result:

Main Theorem: (M. Adams, T. Ratiu, R. Schmid)
The group $FIO_*(M)$ of invertible Fourier integral operators on a **compact** manifold M is a graded ∞-dim ILH-Lie group with graded ∞-dim Lie algebra $\Psi DO(M)$ of all pseudodifferential operators on M.
$FIO_*(M)$ is and ∞-dim principal fiber bundle over the base manifold $Diff^s_\theta(\dot{T}^*M)$ of contact transformations of $\dot{T}^*M$ with gauge group $\Psi DO_*(M)$ of invertible pseudodifferential operators.