LECTURE 5

NON-COMPACT MANIFOLDS
and APPLICATIONS

KdV equation and the group of Fourier integral operators

Korteweg deVries (KdV) equation $u_t = 6uu_x - u_{xxx}$

Poisson bracket $\{F, G\}(u) = \int \frac{\delta F}{\delta u} \partial_x \frac{\delta G}{\delta u} dx$

Hamiltonian $H(u) = \int (u^3 + \frac{1}{2} u_x^2) dx$

Hamilton's equations $u_t = \{u, H\} \Longleftrightarrow u$ satisfies KdV

Gardner, Kruskal 1971

Theorem: (*M.Adams, J.Eichhorn, T.Ratiu, R.Schmid*)
A: The KdV equation is a Hamiltonian system with respect to the Lie-Poisson bracket on the coadjoint orbit of the Lie group of invertible Fourier integral operators $\mathcal{G} = FIO_*$ through the Schrödinger operator.

B: The Kostant-Symes theorem applied to a splitting of the Lie algebra of FIO_* , the space of pseudodifferential operators $\mathfrak{g} = \Psi DO$ gives the complete integrability of KdV, i.e. the Gelfand-Dikii family of commuting integrals, including H.

Kostant-Symes theorem:
Suppose we have a vector space decomposition of the Lie algebra $\mathbf{g}$ into a direct sum of two subalgebras, $\mathbf{g} = \mathbf{h} \oplus \mathbf{k}$. This gives the corresponding decomposition $\mathbf{g}^* = \mathbf{k}^\perp \oplus \mathbf{h}^\perp$ which allows us to identify the duals $\mathbf{h}^* \cong \mathbf{k}^\perp$ and $\mathbf{k}^* \cong \mathbf{h}^\perp$. We obtain functions in involution as follows: Let $F, H : \mathbf{g}^* \to \mathbf{R}$ be two functions that are constant on coadjoint orbits of G in $\mathbf{g}^*$. Then for $A \in \mathbf{h}^*, \{F_A, H_A\} = 0$, where F_A and H_A are the restriction of F and H to the coadjoint orbit of A in $\mathbf{h}^*$.

For KDV the Lie group is $G = FIO_*(S^1)$ with Lie algebra $\mathbf{g} = \Psi DO(S^1)$.
Consider $M = S^1$ the unit circle. Then each pseudodifferential operator $P \in \Psi DO_m(S^1)$has total symbol of the form $p(x,\xi) = \sum_{-\infty<j\leq m} p_j(x)\xi^j$.

The Lie algebra $\mathbf{g} = \Psi DO$ decomposes into the two subalgebras $\mathbf{h} = \Psi DO_- = \cup_{m<0}\Psi DO_m$ and $\mathbf{k} = \Psi DO_+ = \cup_{m\geq 0}\Psi DO_m$,i.e. $\mathbf{g} = \mathbf{h} \oplus \mathbf{k}$
$\Psi DO = \Psi DO_- \oplus \Psi DO_+$

The inner product $< P, Q >= Trace(P \cdot Q)$ where $Trace(P) = \int p_{-1}(x)dx$ identifies $\Psi DO^* \simeq \Psi DO$ and $(\Psi DO_-)^* \simeq \Psi DO_+$

So for $\mathbf{g} = \mathbf{h} \oplus \mathbf{k}$ we get $\mathbf{g}^* = \mathbf{k}^\perp \oplus \mathbf{h}^\perp$, i.e

$$\mathbf{g}^* = \Psi DO^* \simeq \Psi DO_+^\perp \oplus \Psi DO_-^\perp \simeq \Psi DO_-^* \oplus \Psi DO_-^\perp \simeq$$
$$\simeq \Psi DO_+ \oplus \Psi DO_-^\perp$$

The Lie-Poisson bracket on $\mathbf{h}^* = \Psi DO_-^* \simeq \Psi DO_+$ at $A \in \Psi DO_+$ is

$$\{F, H\}(A) =< A, [\frac{\delta F}{\delta A}, \frac{\delta H}{\delta A}] >= \int (A \circ [\frac{\delta F}{\delta A}, \frac{\delta H}{\delta A}])_{-1} dx$$

$...)_{-1}$ means order(-1) part of the symbol.

The Lie-Poisson evolution equations $\dot{F} = \{F, H\}$ for any function F on ΨDO_-^* are equivalent to

$$\dot{A} = X_H(A) = ad^*_{\frac{\delta H}{\delta A}}(A) = [\frac{\delta H}{\delta A}, A]_+$$

on $\Psi DO_-^* \simeq \Psi DO_+$, where $...]_+$ means taking only the part in ΨDO_+

For the Schrödinger operator $A \in \Psi DO_+$ with total symbol $a(x,\xi) = a(x) + \xi^2$ the Lie-Poisson bracket of two functions $F, G : \Psi DO^* \to \mathbf{R}$ at A becomes

$$\{F,G\} = \int \frac{\delta F}{\delta a} \partial_x \frac{\delta G}{\delta a} dx$$

which is the Gardner bracket.

For the Hamiltonian $H = \int (a^3 + \frac{1}{2} a_x^2) dx$ Hamiltons equations $\dot{A} = [\frac{\delta H}{\delta A}, A]_+$ become

$$a_t = 6 a a_x - a_{xxx}$$

which is the KdV equation.

For the functionals
$H_k(A) = Trace(A^k) = \int (A^k)_{-1} dx, \ k \in \mathbf{N}$ we have

$\frac{\delta H_k}{\delta A} = k A^{k-1}$, hence $[A, \frac{\delta H_k}{\delta A}] = [A, kA^{k-1}] = 0$.

Thus H_k are constant on coadjoint orbits. By Kostant-Symes theorem, restricting the H_k to $\Psi DO_-^* \simeq \Psi DO_+$ gives the Gelfand-Dikii family of commuting integrals for KdV.

Example:

$H_0 = \int a \ dx$

$H_1 = \int \frac{1}{2}a^2 \ dx$

$H_2 = \int (a^3 + \frac{1}{2}a_x^2) dx \equiv H$ above !

$H_3 = \int (\frac{5}{8}a^4 + \frac{5}{4}aa_x^2 + \frac{1}{8}a_{xx}^2) dx$

etc.

Diffeomorphism groups and FIO for NON-COMPACT manifolds

J. Eichhorn:
"There is exactly one thing that work in the non-compact case: NOTHING"

Example of what's going wrong:
Let M , N be compact manifolds, then $f : M \to N$ is of Sobolev class $H^s \Longleftrightarrow$ the local representatives
$f^i_j : U_i \subset \mathbf{R}^m \to V_j \subset \mathbf{R}^n$ are of class H^s, where $M = \bigcup(U_i, \phi_i),\ N = \bigcup(V_j, \Psi DOi_j),\ f^i_j := \Psi DOi_j \circ f \circ \phi_i^{-1}$. These covers are **finite** if M, N are compact.
This definition is invariant $\Leftrightarrow$ $s > \frac{n}{2} + 1$.
In the compact case we can define the distance by

$$d^s(f, g) := (\sum_{i,j} \|f^i_j - g^i_j\|^2_s)^{\frac{1}{2}}$$

This definition is meaningless if M, N are open !

Idea: Bounded Geometry

- Control over the metric and its derivatives
- Control over the mappings and their derivatives by the metric i.e. maps adapted to the bounded geometry

Definition: (M^n, g) has *bounded geometry of order* k, $0 \leq k \leq \infty$, if it has a positive injectivity radius and the curvature and all is derivatives up to order k are uniformly bounded; i.e the following two conditions are satisfied:

$$(I): \; r_{inj}(M) = \inf_{x \in M} r_{inj}(x) > 0$$

$$(B_k): \; |\nabla^i R| \leq C_i \;, \; 0 \leq i \leq k.$$

$(I) \Leftrightarrow$ there exists a ball around 0 in $\mathbf{R}^n$ which is domain of normal (geodesic) coordinates **for all** $x \in M$.

$(B_k) \Leftrightarrow$ there exists a constant d_k (independent of $x \in M$) such that $\|g_{ij}\|_{C^k} \leq d_k$ in any normal coordinate system
$\Leftrightarrow \|\Gamma_{ij}^m\|_{C^{k-1}} \leq d_k$ in any normal coordinate system

Examples of manifolds with bounded geometry: compact manifolds, Lie groups, homogeneous spaces, covering spaces of Riemannian manifolds, leaves of foliations of compact manifolds.

Fact: There is **no** topological obstruction for the existence of a complete Riemannian metric with bounded geometry of any order.

Bounded maps $C^{\infty,r}(M,N)$:
Let $(M,g),(N,h)$ open, complete Riemannian manifolds satisfying (I) , (B_k) and $f \in C^{\infty}(M,N)$. Assume $r \leq k$. Let $C^{\infty,r}(M,N)$ be the set of all $f \in C^{\infty}(M,N)$ satisfying

$$|df|_r = \sum_{i=0}^{r-1} \sup_{x \in M} |\nabla^i df|_x < \infty.$$

Equivalently: $f \in C^{\infty,r}(M,N) \Leftrightarrow \frac{\partial^\alpha}{\partial x^\alpha} f^\nu$ is uniformly bounded in any normal coordinate system; $|\alpha| \leq r, 1 \leq r \leq k$.

Topology: $C^{p,r}(M,N)$ the completion of $C^{\infty,r}(M,N)$ in H^p Sobolev topology is a C^{k+1-r}-Banach manifold, $p=2$ Hilbert manifold. f,g are close iff ex. a vector field ξ along f with small Sobolev norm $\|\xi\|_{p,r} < \varepsilon$ such that $g(x) = exp_{f(x)}\xi(x)$

The bounded diffeomorphism group $Diff^{p,r}(M)$

Problem: f bounded $\not\Rightarrow f^{-1}$ bounded, i.e no group
We need an additional assumption then $Diff^{p,r}(M)$ is open in $C^{p,r}(M,M)$ and hence a C^{k+1-r}-Banach manifold, and for $p=2$ it is a Hilbert manifold.

Additional assumption to obtain a group: Set $Diff^{p,r}(M) := \{f \in C^{p,r}(M,M) \mid f \; bijective, \; preserves\ orientation\ and\ |\lambda|_{min}(df) > 0\}$, then $Diff^{p,r}(M)$ is open in $C^{p,r}(M,M)$, hence a C^{k+1-r}-Banach manifold

Theorem:(J. Eichhorn, R. Schmid)
Let (M^n, g) be an open, oriented, complete Riemannian manifold satisfying (I), (B_∞) and let $r > \frac{n}{p} + 1$. Then $Diff^{p,\infty}(M) = \lim_{\leftarrow} Diff^{p,r}(M)$ is an ILB - Lie group; and for $p = 2$ it is an ILH - Lie group.

Volume preserving and symplectic diffeos.

Let ω be a C^∞-bounded non-degenerate q-form, $q = n$ or $q = 2$, let $Diff^{p,r}_\omega = \{f \in Diff^r | f^*\omega = \omega\}$.

Theorem: (Eichhorn , Schmid)
a) $Diff^{p,\infty}_\omega = \lim_{\leftarrow r} Diff^{p,r}_\omega$ is an ILH-Lie group with Lie algebra consisting of divergence free ($q = n$), or locally Hamiltonian ($q = 2$) vector fields ξ with finite Sobolev norm $|\xi|_{p,r}$ for all r.
b) $Diff^{p,r}_\omega$ is an infinite dim. Riemannian manifold, with (weak) metric

$$g(X,Y)_{id} = \int_M (X,Y)_x dvol_x(g)$$

Contact transformations on $\dot{T}^*M$

If (M^n, g) is an open, oriented, complete Riemannian manifold satisfying (I), (B_k) then the Sasaki metric on the co-sphere bundle in $\dot{T}^*M$ satisfies (I), (B_{k-1})
Let θ be the canonical 1-form on T^*M and consider

$$Diff_\theta^{p,r}(\dot{T}^*M) = \{f \in Diff^{p,r}(\dot{T}^*M) |\ f^*\theta = \theta\ \}$$

Theorem: (Eichhorn, Schmid)

$$Diff_\theta^{p,\infty}(\dot{T}^*M) = \lim_{\infty \leftarrow r} Diff_\theta^{p,r}(\dot{T}^*M)$$

is an ILH-Lie group.

This is the space of phase functions for the Fourier integral operators !

Pseudodifferential operators and Fourier integral operators on open manifolds

If (M^n, g) is **open** the previous definition of ψs and $FIOs$ does **not** make sense. We need to adapt the class of symbols and phase functions to the bounded geometry of M in order to obtain globally defined Fourier integral operators $A : C_c^\infty(M) \rightarrow \mathcal{D}'(M)$. Then the corresponding spaces ψ and FIO have similar properties as in the compact case and we can use the same ideas as before to construct Lie group structures.

$$FIO \quad Au(x) = (2\pi)^{-n} \int \int e^{i\varphi(x,y,\xi)} a(x,\xi) u(y) dy\, d\xi$$

- **symbols:** the family of local symbols $a(x, \xi)$ together with their derivatives should be uniformly bounded

- **phase functions:** the phase functions $\varphi(x, y, \xi)$ should locally generate canonical transformations in the space $Diff_\theta^{p,r}(\dot{T}^* M)$

This defines the class of **uniform** pseudodifferential- and Fourier integral operators $\quad \mathcal{U}\Psi DO_m, \quad \mathcal{U}FIO_m$

Again we get an exact sequence of groups

$$I \rightarrow (\mathcal{U}\Psi DO_0)_* \overset{j}{\hookrightarrow} (\mathcal{U}FIO_0)_* \xrightarrow{p} Diff_\theta^{p,r}(\dot{T}^*M) \rightarrow id$$

Now we follow the same ideas as in the compact case: step 1,2...7 to construct ILH Lie groups structures on these spaces.

Main Theorem: (Eichhorn, Schmid, 2001)

$$\begin{aligned} \mathcal{U}\Psi DO &= \lim_{\infty \leftarrow s} \mathcal{U}\Psi DO^s \text{ is an ILH Lie group} \\ \mathcal{U}FIO &= \lim_{\infty \leftarrow t} \mathcal{U}FIO^t \quad \text{is an ILH Lie group} \end{aligned}$$

multiplication $\mu : \mathcal{U}FIO^{t+r} \times \mathcal{U}FIO^t \rightarrow \mathcal{U}FIO^t$
$\mu(A,B) = AB$ is C^k differentiable, $k = min(r,t)$

inversion $\nu : \mathcal{U}FIO^{t+r} \rightarrow \mathcal{U}FIO^t$; $\nu(A) = A^{-1}$

is C^k differentiable, $k = min(r,t)$

right mult. $R_A : \mathcal{U}FIO^t \rightarrow \mathcal{U}FIO^t$: $R_A(B) = BA$

is C^t differentiable, for any $A \in \mathcal{U}FIO^t$

left mult. $L_A : \mathcal{U}FIO^t \rightarrow \mathcal{U}FIO^t$: $L_A(B) = AB$

is C^0 (continuous), for any $A \in \mathcal{U}FIO^t$

APPLICATIONS to HYDRODYNAMICS

1. Euler equations and $Diff_{\mu}^{\infty,r}(M)$

Topological Euler equations

$$E_{top} \left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) & = & grad\ p \\ div_{\mu}\, u(t) & = & 0 \end{array} \right.$$

$\mu =$ fixed volume form on (M^n, g)
$u = u(x,t)$ time dep. C^1 vector field on (M^n, g)
$\nabla = \nabla^g$ the Riemannian covariant derivative
but $div = div_{\mu}$, defined by $L_X \mu = (div_{\mu} X)\mu$.
Then $u(t)$ defines a 1-parameter family of diffeomorphisms f_t defined by

$$\frac{df_s}{ds}|_{s=t} \ = \ u(t) \circ f_t \ .$$

Theorem: (J.Eichhorn, R. Schmid, 2001)
Assume (M^n, g) with (I) and (B_k). Then $u(t)$ satisfies the topological Euler equations E_{top} iff $\{f_t\}_t$ is a geodesic in $Diff_{\mu}^{\infty,r}(M)$.

Classical Euler equations for an incompressible, homogeneous fluid without viscosity
$\nabla = \nabla^g,\ div = div_{dvol_x(g)}$.

Non-homogeneous Euler equations

Let $\rho(x,t) > 0$ be a mass density

$$E_{NH}\ \begin{cases} \frac{\partial u}{\partial t} \ + \ \nabla_{u(t)} u(t) \ = \ \frac{1}{\rho}\ grad\ p \\[2ex] \frac{\partial \rho}{\partial t} \ + \ grad\ \rho \cdot u \ = \ 0 \\[2ex] \qquad\quad div_{\mu}\, u(t) \ = \ 0 \end{cases}$$

$\rho =$ constant $\Rightarrow$ homogeneous Euler equations

for $\mu = \mu(g)$ the corresponding equations on $Diff^s_{\mu}(M)$ are **not** right invariant, i.e. they are not derivable from Arnold's method
Take $\tilde{\mu} = \rho_o \mu(g)$ as volume form, then

Theorem: (Eichhorn, Schmid, 2002)
$u(t)$ is a solution of $(E_{NH}) \Leftrightarrow (f_t)_t =$ flow of $u(t)$ is a geodesic on $Diff^s_{\tilde{\mu}}(M)$ and $\rho(x,t) = \rho_o(f_t^{-1}(x))$.

proof: need generalized Hodge decomposition theorem with densities for open manifolds

KdV equation and the Lie group $\mathcal{U}FIO_*$

Theorem: (J.Eichhorn, R.Schmid, 2003)
The non-periodic KdV equation on the real line is a Hamiltonian system with respect to the Lie-Poisson bracket on the coadjoint orbit through the Schrödinger operator of the infinite dimensional Lie group of invertible Fourier integral operators $\mathcal{G} = \mathcal{U}FIO_*(\mathbf{R})$

ΨDO and quantization

Another interpretation of the exact sequences of Lie groups and Lie algebras

$$I \longrightarrow (\Psi DO_0)* \overset{j}{\hookrightarrow} (FIO_0)* \xrightarrow{p} Diff_\theta^\infty(\dot{T}^*M) \longrightarrow e$$

$$0 \longrightarrow \Psi DO_0 \overset{j}{\hookrightarrow} \Psi DO_1 \xrightarrow{\pi} C^\infty_{+1}(\dot{T}^*M) \longrightarrow 0$$

For $P \in \Psi DO_1$ its principal symbol $\pi(P) : \dot{T}^*M \to \mathbf{R}$ is a smooth function , homogeneous degree $+1$, i.e $\pi(P) \in C^\infty_{+1}(\dot{T}^*M)$, moreover π is a surjective Lie algebra homomorphism

$$\pi([P,Q]) = \frac{1}{i}\{\pi(P), \pi(Q)\}$$

Quantization of $C^\infty_{+1}(\dot{T}^*M)$ via ΨDO_1 !

Consider the Lie subalgebra $DO_1 \subset \Psi DO_1$ of all differential operators of order 1. $DO_1 \cong \mathcal{X}^\infty(M)$ smooth vector filed on M. i.e. we consider a vector field X on M as ΨDO of order 1
locally $X = \sum X^j(x)\frac{\partial}{\partial x_j} \Rightarrow \pi(X) = \sum X^j \xi_j$ and

$$Xu(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} \pi(X) u(y) dy d\xi$$

consider Lie subalgebra $L(T^*M) \subset C^\infty_{+1}(\dot{T}^*M)$ of all smooth functions on T^*M linear on each fiber T^*_xM. Locally $f \in L(T^*M)$, $f(x,\xi) = \sum f^j(x)\xi_j$

Theorem: The symbol map π induces a Lie algebra isomorphism $\pi : \mathcal{X}^\infty(M) \to L(T^*M)$:
$\pi(X) \cdot \alpha_x = i\alpha_x \cdot X(x), x \in M, \alpha_x \in T^*_xM$

$$\pi([X,Y]) = \frac{1}{i}\{\pi(X), \pi(Y)\}$$

Quantization of $L(T^*M)$ via $\mathcal{X}^\infty(M)$!

General: For any $f, g \in C^\infty(T^*M)$ define $P, Q \in \Psi DO$ such that P has principal symbol f and Q has principal symbol g. Then the principal symbol of $[P,Q]$ if $\{f,g\}$.

References:*

M.R. Adams, T.S. Ratiu and R. Schmid, *The Lie Group Structure of Diffeomorphism Groups and Invertible Fourier Integral Operators with Applications*, MSRI Publication #4, V. Kac (ed.) (1985), 19-41.

M.R. Adam, T.S. Ratiu, and R. Schmid, *A Lie Group Structure for Pseudodifferential Operators*, Math. Ann. **273** (1986), 529-551 .

M.R. Adam, T.S. Ratiu, and R. Schmid, *A Lie Group Structure for Fourier Integral Operators*, Math. Ann. **276** (1986), 19-41.

A. Banyaga , *The Structure of Classical Diffeomorphism Groups*, Kluwer Acad. Publ, Dortrecht, 1997.

J. Eichhorn, and R. Schmid *Form Preserving Diffeomorphisms on Open Manifolds*, Ann. Global Analysis and Geometry **14** (1996), 147-176.

J. Eichhorn, and R. Schmid, *Lie Groups of Fourier Integral Operators on Open Manifolds*, Comm. Analysis and Geometry, **9** (2001), 983-1040.

H. Omori, *Infinite - Dimensional Lie Groups*, Transl. Math. Monogr. **158**, AMS, Providence, 1979.

A. Pressley, and G. Segal *Loop Groups*, Oxford Univ. Press, Oxford 1986.

T.S. Ratiu, and R. Schmid, *The Differentiable Structure of Three Remarkable Diffeomorphism Groups*, Math. Zeitschrift **177** (1981), 81-100.

R. Schmid, *Infinite Dimensional Hamiltonian Systems* ,Monographs and Textbooks in Physical Sciences, Biblionopolis, Napoli, 1987.

* More detailed references can be found in the papers above.