Infinite Dimensional Lie Groups in Mathematical Physics

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A) Some mathematical problems with infinite dimensional Lie groups

B) Examples and physical applications

Lie group G $\nearrow$ group, $\searrow$ manifold $\Big\}$ compatible: group multiplication & inversion are smooth

multiplication $\mu : G \times G \to G \quad \mu(g,h) = g \cdot h \quad C^\infty$

inversion $\nu : G \to G \quad \nu(g) = g^{-1} \quad C^\infty$

manifold G

$\nearrow$ finite dimensional $\Longrightarrow$ locally $\mathbb{R}^n$, $(\mathbb{C}^n)$

$\searrow$ ∞ dimensional $\Longrightarrow$ locally ∞ dimensional vector space

- Hilbert space , $< \cdot, \cdot >$ inner product
- Banach space , $\| \cdot \|$ norm
- Fréchet space , $d(\cdot,\cdot)$ metric (no norm)

Lie algebra of G : $\mathfrak{g} = T_e G \simeq$ left invariant vector fields on G

$$L_g^* X = X$$

$$\xi \in \mathfrak{g} \mapsto X_\xi(g) := T_e L_g(\xi) \quad \text{left invariant}$$

Lie bracket : $[\xi, \eta] := [X_\xi, X_\eta](e)$ $\begin{cases} \bullet \text{ bilinear} \\ \bullet \text{ skew symmertic} \\ \bullet \text{ Jacobi identity} \end{cases}$

Examples: Finite dimensions:

matrix Lie groups G : GL(n), SL(n), O(n), SO(n), Sp(2n), U(n), SU(n)

$\mu(A, B) = AB$ matrix multiplication,
$\nu(A) = A^{-1}$ matrix inversion, $e = \mathbb{I}$

matrix Lie algebras $\mathfrak{g}$: L(n), $\mathfrak{sl}(n)$, $\mathfrak{o}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$

$[A, B] = AB - BA$ commutator bracket

Classical results in **finite** dimensions which are **not true** in infinite dimesions

1) Exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism

$\Rightarrow$ canonical coordinates

$\xi \in \mathfrak{g} \mapsto X_\xi$ left invariant vector field $\to \Phi_\xi(t)$ flow of $X_\xi \Rightarrow \exp(\xi) := \Phi_\xi(1)$

For matrix groups $\exp(A) = e^A = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$

2) $f_1, f_2 : G \to H$ smooth Lie group homomorphisms $f_i(g \cdot h) = f_i(g) \cdot f_i(h)$
(G connected) Then $T_e f_1 = T_e f_2 \Rightarrow f_1 = f_2$

3) $f : G \to H$ continuous homomorphism $\Rightarrow$ f analytic

4) $H \subset G$ closed subgroup $\Rightarrow$ H Lie subgroup (Lie group and submanifold)

5) For G any Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra
$\exists!$ connected Lie subgroup $H \subset G$ with $\mathfrak{h}$ its Lie algebra, $T_e H \simeq \mathfrak{h}$

6) For $\mathfrak{g}$ any **finite** dimensional Lie algebra there exists a connected
Lie group G with $T_e G \simeq \mathfrak{g}$

Examples: Infinite dimensions:

1) Let M be a finite dimensional manifold and let
$\mathbf{G} = \mathbf{C}^{\infty}(\mathbf{M}, \mathbb{R})$, smooth functions on M, $\mu(f, g) = f + g$, $\nu(f) = -f$, $e = 0$

Lie algebra $\mathfrak{g} = T_0 C^{\infty}(M, \mathbb{R}) \simeq C^{\infty}(M, \mathbb{R})$, $[\xi, \eta] = 0$, abelian , $\exp = \mathrm{id}$

completion: C^k -norm ($k < \infty$) $\Rightarrow$ Banach Lie group $C^k(M, \mathbb{R})$
H^s - Sobolev norm $\Rightarrow$ Hilbert Lie group $H^s(M, \mathbb{R})$

Application: Maxwell's equations

$$\dot{B} = -\nabla \times E \,, \qquad \nabla \cdot E = \rho$$
$$\dot{E} = \nabla \times B \,, \qquad \nabla \cdot B = 0$$
$$\Downarrow$$
$$B = \nabla \times A, \text{ vector potential } A$$

gauge invariance: $G = C^{\infty}(\mathbb{R}^3)$ gauge group
gauge transfromation

$$f \in G : A \mapsto f \cdot A := A + \nabla f$$

leaves Maxwell's equations invariant: **Gauge Symmetry**

2) $G = C^\infty(M, \mathbb{R} \setminus \{0\})$, $\mu(f,g) = f \cdot g$, $\nu(f) = f^{-1}$, $e = 1$

C^k -topology: $C^k(M, \mathbb{R} \setminus \{0\})$ open in $C^k(M, \mathbb{R})$

M **compact** $\Rightarrow C^k(M, \mathbb{R} \setminus \{0\})$ Banach Lie group

H^s Sobolev-topology:
M **compact** $\Rightarrow H^s(M, \mathbb{R} \setminus \{0\})$ closed under multiplication
$s > \frac{1}{2} \dim M \Rightarrow H^s(M, \mathbb{R} \setminus \{0\})$ Hilbert Lie group

3) generalize : $\mathbb{R} \setminus \{0\} \rightsquigarrow \mathrm{G}$ finite dim. Lie group

$\mathcal{G} = C^k(M, \mathrm{G})$, $\mu(f,g)(x) = f(x) \cdot g(x)$, $\nu(f) = f^{-1}$, $e = 1$

$\mathfrak{G} = C^k(M, \mathfrak{g})$, $[\xi, \eta](x) = [\xi(x), \eta(x)]$, bracket in $\mathfrak{g}$

Applications: Gauge theories, quantum field theory, loop groups

Special case $M = S^1$ **circle** : $\mathcal{G} = C^k(S^1, \mathbf{G}) = L^k(\mathbf{G})$ loop group

$\mathfrak{G} = C^k(S^1, \mathfrak{g})$ loop algebra $\longrightarrow$ affine algebra,
Kac-Moody Lie algebras (central extensions)

Exponential map is local diffeomorphism

$$\mathrm{Exp} : C^k(S^1, \mathfrak{g}) \to C^k(S^1, \mathbf{G}) : \mathrm{Exp}(\xi) = \exp \circ \, \xi$$

where $\exp : \mathfrak{g} \longrightarrow \mathbf{G}$ finite dim.

Applications:

- completely integrable systems, soliton equations (KdV)
- Toda, KdV, KP hierarchies
- Quantum field theories , Lie algebra cohomology $\leftrightarrow$ **BRST symmetry** (Schmid 90's)

Ex. of loop algebra that is not the Lie algebra of any Lie group
(irrational co-cycle condition)

4) **Diffeomorphism groups**

M **compact** manifold

$$\mathrm{Diff}(M) = \{f : M \to M \mid f \text{ diffeomorphism}\} \quad \mathbf{C^k,\ C^\infty,\ H^s}$$

Algebra of Diff(M)? group: $\mu(f,g) = f \circ g,\ \nu(f) = f^{-1},\ e = id_M$

cohomology, isomorphy types , ... (Banyaga 1970's ...,1997)

Geometry of Diff(M)? topology, differential structure
global analysis

$\mathrm{Diff}^k(M)$ Banach manifold (Palais 68, Omori 74)
$\mathrm{Diff}^\infty(M)$ Fréchet manifold (Omori, Gutknecht, Schmid, 78)
$\mathrm{Diff}^s(M)$ Hilbert manifold (Ebin-Marsden-Fischer,Ratiu-Schmid 79)

Same for **non-compact** manifolds, Eichhorn-Schmid 96

Theorem: (Marsden , Ratiu- Schmid 70's, Eichhorn-Schmid 96)

$$\mu : \mathrm{Diff}^{s+k}(M) \times \mathrm{Diff}^{s}(M) \to \mathrm{Diff}^{s}(M) \quad \text{is} \quad C^k, \qquad k = 0 \quad \mu \text{ continuous}$$

$$\nu : \mathrm{Diff}^{s+k}(M) \to \mathrm{Diff}^{s}(M) \quad \text{is} \quad C^k, \qquad k = 0 \quad \nu \text{ continuous}$$

$$\mathrm{Diff}^{\infty}(M) = \lim_{\leftarrow k} \mathrm{Diff}^{k}(M) \ , \quad \mathbf{ILB} \ - \ \mathbf{Lie\ group}$$

$$\mathrm{Diff}^{\infty}(M) = \lim_{\leftarrow s} \mathrm{Diff}^{s}(M) \ , \quad \mathbf{ILH} \ - \ \mathbf{Lie\ group}$$

"Fréchet Lie groups"

Lie algebra: $\mathfrak{g} = T_e\mathrm{Diff}^{\infty}(M) = \mathfrak{X}^{\infty}(M)$ vector fields on M
Lie algebra only for C^{∞}vector fields
[X,Y]=XY-YX commutator bracket

Exponential map: $\mathrm{Exp} : \mathfrak{g} = \mathfrak{X}(M) \to G = \mathrm{Diff}(M) : \ \mathrm{Exp}(X) = \Phi_{t=1}$,
flow Φ_t of X at time $t = 1$.
Exp is **not** a local diffeomorphism, it is **not** locally onto

General Construction:

M, N finite dimensional smooth manifolds (M compact $\lesssim$ 1996)

$C^\infty(M,N)$, $C^k(M,N)$, $C^s(M,N)$ **are smooth manifolds**
$\mathrm{Diff}^\infty(M) \subset C^\infty(M,M)$ **open submanifold**, same for C^k, C^s

Application $C^\infty(M,N)$**:** Quantum field theory , σ-model, branes

Applications of Diff(M) (classical)

1) **General relativity:** M Lorentz 4-manifold , g metric

Ric(g)=0 Einstein's field equations

they are invariant under Diff(M) (coordinate transformations)
Ric(g)=0 is an ∞-dim Hamiltonian system on the space (Fischer-Marsden)

$$\mathcal{M} = \{\mathbf{metrics}\}/\mathbf{Diff(M)}$$ infinite dim. moduli space

These are the Euler - Lagrange equations for the Lagrangian density

$\mathcal{L}(\mathrm{g}) = \mathrm{R}(\mathrm{g})\mu(\mathrm{g})$, R(g) scalar curvature, μ(g) volume

2) **Subgroups of Diff(M)**

μ = volume on M , $\text{Diff}_\mu(M) = \{f \in \text{Diff}(M) \mid f^*\mu = \mu\}$
volume preserving diffeomorphisms

Theorem: (Ebin-Marsden (compact) 70, Eichhorn-Schmid (open)96)
$\text{Diff}_\mu(M)$ is a closed Lie subgroup of Diff(M).

Lie algebra: $\mathfrak{X}_\mu(M) = \{X \in \mathfrak{X}(M) \mid \text{div}_\mu X = 0\}$
divergence free vector fields

$\mathfrak{X}_\mu(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$, [X,Y]=XY-YX commutator bracket

Note: ∞ - dim. $\mathfrak{X}_\mu(M)$ Lie algebra $\not\Rightarrow \exists$ corresponding Lie group
$\text{Diff}_\mu(M) \subset \text{Diff}(M)$ closed $\not\Rightarrow$ Lie subgroup

Applications: Fluid dynamics
Euler's equations $\Leftrightarrow$ geodesics on $\text{Diff}_\mu(M)$

Incompressible fluids

μ = volume on M (μ = n-form, $\mu \neq 0$)

$\mathrm{Diff}^s_\mu(M) = \{f \in \mathrm{Diff}(M) \mid f^*\mu = \mu\}$
= configuration space of incompressible fluid
∞ - dim. Riemannian manifold

Euler's equations for incompressible, homogeneous fluid

$$\frac{\partial u}{\partial t} + \nabla_u u = -\nabla p\ , \quad \mathrm{div}\ u = 0$$

u = velocity field of fluid = vector field on M $\rightsquigarrow \alpha_t$ flow
incompressibility $\Rightarrow \alpha_t \in \mathrm{Diff}^s_\mu(M)$

Theorem: (Ebin-Marsden (compact) 70, Eichhorn-Schmid (open)96)

- u satisfies Euler's equations $\Leftrightarrow \alpha_t$ geodesics on $\mathrm{Diff}^s_\mu(M)$
- existence of C^∞ geodesics on $\mathrm{Diff}^s_\mu(M)$ for small t
- Euler's equation has unique solution for small t, C^∞ in u_0

Generalization: **Topological Euler equations:**

μ = fixed volume form on (M, g), open Riemannian manifold
$u = u(x,t)$ is a time dependent C^1 vector field on (M, g)
$\nabla = \nabla^g$ the Riemannian covariant derivative
but now div $= \mathrm{div}_\mu$, defined by $L_X\mu = (\mathrm{div}_\mu X)\mu$.

$$(\mathrm{Euler}_{\mathrm{top}})\ \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) &= \mathrm{grad}\ p \\ \mathrm{div}_\mu\, u(t) &= 0 \end{cases}$$

Theorem: (Eichhorn-Schmid, 02)
If M has bounded geometry of any order, then $u(t)$ satisfies the topological Euler equations $(\mathrm{Euler}_{\mathrm{top}})$ iff α_t is a geodesic in $\mathrm{Diff}^{\infty,r}_{\mu,0}(M)$. (work in progress)

3) **Other subgroups of Diff(M)**

θ = contact 1-form on M ($\theta \wedge (d\theta)^n$ = volume on M, dim. M = 2n+1)

$\mathcal{G} = \mathrm{Diff}_\theta(M) = \{f \in \mathrm{Diff}(M) \mid f^*\theta = \theta\}$ quantomorphism group
$\subset$ Diff(M) Lie subgroup contact transformations

$\mathfrak{G} = \mathfrak{X}_\theta(M) = \{X \in \mathfrak{X}(M) | L_X\theta = 0\} \subset \mathfrak{X}(M)$ Lie subalgebra

Applications: • quantization (Schmid 91)
• Fourier integral operators (Adams-Ratiu-Schmid 86, Eichhorn-Schmid 02)

4) ω = symplectic 2-form on M (nondegenerate, closed, $d\omega = 0$)

$\mathcal{G} = \mathrm{Diff}_\omega(M) = \{f \in \mathrm{Diff}(M) \mid f^*\omega = \omega\}$ symplectomorphism group
$\subset$ Diff(M) Lie subgroup canonical transformations
$\mathfrak{G} = \mathfrak{X}_\omega(M) = \{X \in \mathfrak{X}(M) | L_X\omega = 0\} \subset \mathfrak{X}(M)$ locallyHamiltonian vector fields

Applications: • classical mechanics: $\omega = \sum dq_i \wedge dp_i$
• **plasma physics**
(Maxwell-Vlasov equations: Marsden-Weinstein-Ratiu-Schmid)

Applications: Plasma physics

The **Maxwell-Vlasov** equations for the plasma density f(x,v,t) and the electric field E(x,t) , magnetic field B(x,t) are the couple system of nonlinear PDEs

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + (E + v \times B)\frac{\partial f}{\partial v} = 0$$

$$\frac{\partial B}{\partial t} = -\mathrm{curl}\ E$$

$$\frac{\partial E}{\partial t} = \mathrm{curl}\ B - J_f, \quad J_f = \mathrm{currentdensity}$$

$$\mathrm{div}\ E = \rho_f, \quad \rho_f = \mathrm{chargedensity}$$

$$\mathrm{div}\ B = 0$$

Theorem:(Marsden-Ratiu-Schmid-Weinstein)
This coupled non-linear system of evolution equations is Hamiltonian $\dot{F} = \{\{F, H\}\}$ on the reduced phase space

$\mathcal{MV} = (\mathrm{T}^*\mathrm{Diff}_\omega(\mathbb{R}^6) \times \mathrm{T}^*\mathcal{A})/\mathrm{C}^\infty(\mathbb{R}^6)$ with respect to the reduced Poisson bracket

$$\{\{F, G\}\}(f, E, B) = \int f\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\}dxdv$$

$$+ \int (\frac{\delta F}{\delta E}\mathrm{curl}\frac{\delta G}{\delta B} - \frac{\delta G}{\delta E}\mathrm{curl}\frac{\delta F}{\delta B})dxdv$$

$$+ \int (\frac{\delta F}{\delta E} \cdot \frac{\partial f}{\partial v}\frac{\delta G}{\delta f} - \frac{\delta G}{\delta E} \cdot \frac{\partial f}{\partial v}\frac{\delta F}{\delta f})dxdv$$

$$+ \int f \cdot B(\frac{\partial}{\partial v}\frac{\delta F}{\delta f} \times \frac{\partial}{\partial v}\frac{\delta G}{\delta f})dxdv$$

with Hamiltonian

$$H(f, E, B) = \frac{1}{2}\int v^2 f(x, v, t)dv + \frac{1}{2}\int (|E|^2 + |B|^2)dx$$

$$\mathrm{Maxwell - Vlasov} \Leftrightarrow \dot{F} = \{\{F, H\}\}$$

Fourier integral operators , KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad \text{KdV equation}$$

Gardner-Kruskal 70's:

$$\{F, G\} = \int_0^{2\pi} \frac{\delta F}{\delta u} \frac{\partial}{\partial x} \frac{\delta G}{\delta u} dx, \quad H(u) = \int_0^{2\pi} (u^3 + \frac{1}{2} u_x^3) dx$$

$$\mathbf{\dot{u} = \{u, H\} \iff u \text{ satisfies KdV}}$$

Where does this come from?

The symmetry group for KdV is $\mathcal{G} = \text{FIO}$ invertible Fourier integral operators on the circle S^1.

Fourier integral operator FIO : $A : C^\infty(M) \to C^\infty(M)$ locally given by

$$A(u)(x) = (2\pi)^{-n} \int\int e^{i\phi(x,y,\xi)} a(x,\xi) u(y) dy d\xi$$

pseudodifferential operator ΨDO : $P : C^\infty(M) \to C^\infty(M)$ locally given by

$$P(u)(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} p(x,\xi) u(y) dy d\xi$$

Theorem: (Adams-Eichhorn-Ratiu-Schmid,86,02)

a) The space of all invertible Fourier integral operators $\mathcal{G} = \mathsf{FIO}$ is a smooth ∞ dimensional ILH - Lie group

b) FIO is a smooth ∞ dimensional principal fiber bundle over the base space $\mathrm{Diff}_\omega(T^*M - \{0\})$ with structure group ΨDO (invertible pseudo differential operators)

c) The Gardner-Kruskal bracket is the Lie Poisson bracket on the coadjoint orbit of FIO through the Schroedinger operator.

KdV is completely integrable with $H = H_2$ in the hierarchy.

Symplectic Algorithms for ∞ dimensional Hamiltonian Systems

Symplectic integrators include a variety of different time-discretization schemes designed to preserve the global symplectic structure of the ∞ -dim. phase space for a Hamiltonian system. They show substantial benefits in numerical computation for Hamiltonian system, especially in large-scale simulations.

Approach:

1) Discretize the equation in time direction to preserve the symplectic structure

2) Approximate the equation in space direction, using finite difference, finite element, or spectral methods.

Examples: (Lu-Schmid) Maxwell's equations, wave equations, Sine-Gordon, Klein-Gordon, Φ^4 - equations.