How do you differentiate in infinite dimensions ?

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Recently a lot of papers and talks appeared about INFINITE dimensional spaces

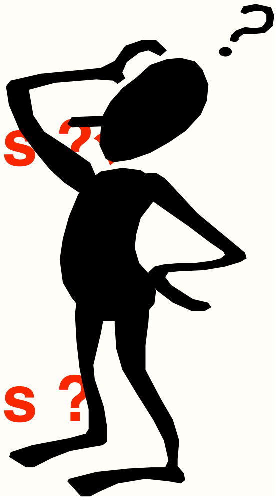
- infinite dimensional manifolds
- infinite dimensional Lie groups
- Hilbert manifolds ,- Lie groups
- Banach manifolds ,- Lie groups
- Fréchet manifolds ,- Lie groups

Finite dimensional manifold locally $\mathbf{R}^n$ **calculus ?** ✓

Hilbert manifold
locally Hilbert space
inner product $< x , y >$
norm $\|x\| = < x , x >^{1/2}$
metric $d(x,y) = \|x-y\|$
calculus ? ✓

Banach manifold
locally Banach space
norm $\|x\|$
metric $d(x,y) = \|x-y\|$
calculus ?

Fréchet manifold
locally Fréchet space
metric $d(x,y)$
calculus ?



It is a well known fact that the differential calculus in $\boldsymbol{R}$ extends canonically to $\boldsymbol{R}^n$ and to Hilbert and Banach spaces.

However, if one wants to develop a theory of differential calculus beyond Banach spaces say to Fréchet spaces one runs into serious difficulties.

That's what I want to explain in this lecture.

What does differentiation of a function *f* in a point *x* mean ?

Generally in analysis one means the approximation of the increase of the function by a *linear* functional of the increment h of the argument, called the *differential* $Df(x)h$

$$f(x+h) - f(x) = Df(x)h + r(h)$$

where one requires that the remainder r(h) goes to zero faster than the differential Df(x)h.

1 -dimensional

$f : \boldsymbol{R} \to \boldsymbol{R}$ is differentiable at $x \in \boldsymbol{R}$ if the limit of the difference quotient exists

(α)
$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =: f'(x) \in R}$$

Avoiding the quotient we get the beginning on the Taylor series

(β)
$$\boxed{f(x+h) = f(x) + f'(x) \cdot h + Rf(x,h)} \qquad \boxed{\lim_{h \to 0} \frac{Rf(x,h)}{h} = 0}$$

remainder condition

Generalize

$f : \boldsymbol{R} \to \boldsymbol{R}^n$ (α) and (β) make sense

$f : \boldsymbol{R} \to \boldsymbol{F}$ (α) , (β) make sense, if F is top. vector space (limit !) $f'(x) \in F$

$f : \boldsymbol{R}^m \to \boldsymbol{R}^n$ (α) no, quotient not possible

(β) no, product $f'(x) \bullet h$ not possible.

New interpretation: multiplication with fixed factor $f'(x)$ is linear map : $\boldsymbol{R} \to \boldsymbol{R}$

(γ)
$$\boxed{f(x+h) = f(x) + Df(x) \cdot h + Rf(x,h)}$$

$Df(x) : R^m \to R^n$ linear

$$\boxed{\lim_{h \to 0} \frac{Rf(x,h)}{\| h \|} = 0 \in R^n} \Leftrightarrow \boxed{\lim_{\| h \| \to 0} \frac{\| Rf(x,h) \|}{\| h \|} = 0 \in R}$$

total (Fréchet) derivative of f at x

(γ) $\boxed{f(x+h) = f(x) + Df(x)\cdot h + Rf(x,h)}$ $\boxed{\lim_{h\to 0} \frac{Rf(x,h)}{\| h \|} = 0}$

makes sense when $f : \boldsymbol{E} \to \boldsymbol{F}$ **Banach spaces** (complete,normed)

Fréchet (1911) Total derivative: Let E, F Banach spaces, $U \subseteq E$ open

Definition: $f : U \subseteq E \to F$ is differentiable at $x \in U$ if there exist a continuous linear map $Df(x) : E \to F$ satisfying **(γ)**

Remarks: 1) completeness not needed , normable vector spaces enough

2) no norm needed in F, top. vector space enough

3) f differentiable at $x \Rightarrow f$ continuous at x

4) Differentiability and value $Df(x)$ independent of (equivalent) norms

Calculus : product rule, chain rule etc $D(g \circ f)(x) = Dg(f(x)) \circ Df(x)$

Inverse Function Theorem: $Df(x)$ isomorphism $\Rightarrow f$ local diffeomorphism

Gâteaux (1913) Directional derivative: E, F Banach spaces, $U \subseteq E$ open

Definition: Let $f : U \subseteq E \to F$, $(x,h) \in U \times E$

The directional derivative of f at x in direction of h is defined as

$$\boxed{\delta f(x,h) := \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}}$$

Property: $\delta f(x,h)$ homogeneous in h, **not** linear in h

Gâteaux - Lévy differentiability:

Addition: $h \to \delta f(x,h) : E \to F$ linear and continuous

There is NO **Gâteaux - Lévy calculus** **No chain rule**

Ex: $f: \mathbf{R} \to \mathbf{R}^2$, $f(t)= (t,t^2)$, $g: \mathbf{R}^2 \to \mathbf{R}$, $g(x,y)= x$ if $y=x^2$, otherwise $g = 0$

f differentiable at 0, g Gâteaux-Levy diff. at 0 with $g'(0)=0$ *but* $g \circ f = id$

so $(g \circ f)'(0) = 1$, *but* $(g \circ f)'(0)= g'(f(0))\cdot f'(0) = 0$

Theorem: (Lévy 1923)

Let $f : U \subseteq E \to F$ Gâteaux- Lévy differentiable at every x $\in$ U and
let $Df : U \to L_b(E,F)$ be continuous (norm-topology) then
f Fréchet is differentiable and $\delta f(x,h) = Df(x)h$

Analogy: $f : \mathbf{R}^n \to \mathbf{R}^m$, if all partial derivatives exist and are continuous, then f is has total derivative

Ex: $f : \mathbf{R}^2 \to \mathbf{R}$: $$f(x,y) = \begin{cases} 1 & \text{if } x = 0 \quad \text{or } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

then $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$ i.e.Gâteaux- Lévy differentiable

but f **not** continuous at (0,0) , f not Fréchet differentiable at 0

General : Topological vector spaces

E, F locally convex topological vector spaces, $U \subseteq E$ open

$\{|\,.\,|_\alpha\}_{\alpha\in A}$, $\{|\,.\,|_\beta\}_{\beta\in B}$ defining families of seminorms in E, F

$\mathcal{L}(E,F) = \{\, f : E \to F \mid f$ linear , continuous$\}$

Ansatz: *f: U ⊆ E → F is differentiable at x ∈ U if*

$$\boxed{f(x+h) = f(x) + Df(x)\cdot h + Rf(x,h)} \qquad \boxed{Df(x) \in \mathcal{L}(E,F)}$$

This Ansatz is basic to all notions of differentiability, only the conditions on $Rf(x,h)$ are different : $r(h) := Rf(x,h) \to 0$ faster than $Df(x)h$

Ex:

❶ $$(\forall\beta)(\exists\alpha) \therefore \lim_{|h|_\alpha \to 0} \frac{|r(h)|_\beta}{|h|_\alpha} = 0$$

❷ $$(\forall\beta)(\exists\alpha) \therefore \lim_{h \to 0} \frac{|r(h)|_\beta}{|h|_\alpha} = 0$$

❸ $$(\exists\alpha)(\forall\beta) \therefore \lim_{|h|_\alpha \to 0} \frac{|r(h)|_\beta}{|h|_\alpha} = 0$$

❹ $$(\exists\alpha)(\forall\beta) \therefore \lim_{h \to 0} \frac{|r(h)|_\beta}{|h|_\alpha} = 0$$

each ⇒ Gâteaux

$$\begin{tikzcd}
\text{❸} \arrow[r, shift right] \arrow[d, shift left] & \text{❶} \arrow[l, dotted, shift right, "\text{E or F norm}"'] \arrow[d, shift right] \\
\text{❹} \arrow[r, shift left] \arrow[u, dotted, shift left, "\text{E norm}"] & \text{❷} \arrow[l, dotted, shift left, "\text{E or F norm}"] \arrow[u, dotted, shift right, "\text{E norm}"']
\end{tikzcd}$$

all equivalent if E norm

Fréchet spaces

Definition: *A* Fréchet space is a complete metrizable topological vector space

Topology defined by a family of seminorms $\{|\,.\,|_\alpha\}_{\alpha \in A}$

Examples: a) $\boldsymbol{R}^n$, Hilbert spaces, Banach spaces,
b) Function spaces

Let $U \subset \boldsymbol{R}^n$ open, $C^\infty(U,\boldsymbol{R}) := \{ f : U \to \boldsymbol{R} \mid f \text{ smooth}, C^\infty \}$ Fréchet

seminorms: let $K \subset U$ *compact*, $r \in \boldsymbol{N}$

$$|f|_{K,r} := \max_{0 \le k \le r} \sup_{x \in K} \| D^k f(x) \|$$

countable family

where $\|u\| := \sup_{\substack{\|x_i\| \le 1 \\ 1 \le i \le k}} |u(x_1, x_2, \ldots, x_k)|$ is the norm of $u \in L^k(\boldsymbol{R}^n, \boldsymbol{R})$

- similar for *E, F finite dim. vector spaces* $U \subset E$ *open* : $C^\infty(U,\boldsymbol{F})$ Fréchet

- *vector bundle* (B,π,M), $U \subset M$ open, smooth sections $\Gamma^\infty(U,B)$ Fréchet

Functions of class C^1, C^p, C^∞

continuously differentiable functions

Let $f: U \subseteq E \to F$ be differentiable at $x \in U$, i.e

$$f(x+h) = f(x) + Df(x) \cdot h + Rf(x,h) \quad \text{with} \quad Df(x) \in \mathcal{L}(E,F)$$

f is C^1 if $Df : U \subseteq E \to \mathcal{L}(E,F)$ continuous, need topology on $\mathcal{L}(E,F)$

E norm ⇒ canonical Fréchet differentiability $\quad \lim_{h \to 0} \frac{r(h)}{\| h \|} = 0$

E and F norm ⇒ norm topology on $\mathcal{L}(E,F)$ $\quad \| u \| := \sup_{\| x \| \leq 1} \| u(x) \|$

Conclusion: we need two things for C^1

1) remainder condition r(h) → 0

2) topology on $\mathcal{L}(E,F)$

C^p **$D^p f : U \subseteq E \to \mathcal{L}^p(E,F)$ continuous, $D^0 f = f$, $D^p f = D(D^{p-1} f)$**

1. Question : does f C^1 $\Rightarrow$ $(x,h) \to Df(x)h : E \times E \to F$ continuous ?

Suff. condition: evaluation $ev : \mathcal{L}(E,F) \times E \to F : ev(u,h) = u(h)$ continuous

then
$$\begin{tikzcd} U \times E \arrow[r, "Df \times id"] & \mathcal{L}(E,F) \times E \arrow[r, "ev"] & F \\ (x,h) \arrow[r, "Df \times id"] & (Df(x),h) \arrow[r, "ev"] & Df(x)h \end{tikzcd}$$

YES if E and F normable

Theorem: For any locally convex topology on $\mathcal{L}(E,F)$

$ev : \mathcal{L}(E,F) \times E \to F$ is continuous $\Leftrightarrow$ E normable

Corollary: E NOT normable $\Rightarrow$ ex. no topology on $\mathcal{L}(E,F)$

such that $ev : \mathcal{L}(E,F) \times E \to F$ is continuous

2. Question: does $E \xrightarrow{f} F \xrightarrow{g} G$ C^1 $\Rightarrow$ $g \circ f : E \to G$ C^1 ?

Suff. cond: composition $c : \mathcal{L}(F,G) \times \mathcal{L}(E,F) \to \mathcal{L}(E,G) : c(g,f) = g \circ f$ continuous

Then by chain rule: $D(g \circ f)(x) = Dg(f(x)) \circ Df(x)$

YES if E and F normable, otherwise NO

Answers

- convergence structures on $\mathcal{L}(E,F)$
- limit structures on $\mathcal{L}(E,F)$
- pseudo topologies on $\mathcal{L}(E,F)$

Example: The filters $\mathcal{F}$ on $\mathcal{L}(E,F)$ which converge to 0 are characterized as follows

$\{|\,.\,|_\alpha\}_{\alpha\in A}$, $\{|\,.\,|_\beta\}_{\beta\in B}$ seminorms in *E, F*

$$(\forall\beta)(\exists\alpha)(\forall\varepsilon>0)(\exists A\in\mathcal{F})\ \sup_{u\in A}\ \sup_{|x|_\alpha\leq 1}|u(x)|_\beta\leq\varepsilon$$

Remark: R. Hamilton assumes (x,h)→Df(x)h continuous

⇔ Putting continuous convergence structure Λ_c on $\mathcal{L}_c(E,F)$ (coarsest such that ev continuous *)*

Theorem: $g: E\rightarrow\mathcal{L}_c(E,F)$ continuous ⇔ $g^*: E\times E\rightarrow F$ continuous

$g^* = ev\circ(g,id)$

$$\begin{tikzcd}
(KE) & C^p_{\Theta} \arrow[l] \arrow[d, shift left] & C^p_{\Gamma} \arrow[l, "X" marking] \arrow[d, shift left] & \\
(HL) & C^p_{\Pi} \arrow[l] \arrow[u, dashed, shift left, "\text{E or F norm}"] \arrow[r, dashed, shift left, "\text{E or F norm}"] \arrow[d, shift left] & C^p_{\Delta} \arrow[l, shift left] \arrow[u, dashed, shift left, "\text{E norm}"'] & \\
(FB) & C^p_{qb} \arrow[l] \arrow[u, dashed, shift left, "\text{E norm or Schw.}"] \arrow[r, shift right] \arrow[d, shift left] & C^p_{b} \arrow[l, dashed, shift right, "\text{E Schw.}"'] \arrow[r] \arrow[d, shift left, "\text{E Schw.}"] & (G_{b}) \\
(MB) & C^p_{c} \arrow[l] \arrow[u, dashed, shift left, "\text{E Schw.}"] \arrow[r, shift right] & C^p_{pk} \arrow[l, dashed, shift right, "\text{E metr.}"'] \arrow[u, dashed, shift left] \arrow[r] \arrow[d, shift left] & (G_{pk}) \\
 & & C^p_{k} \arrow[u, dashed, shift left, "\text{E compl. or metr.}"] \arrow[r] \arrow[d, shift left] & (G_{k}) \\
 & & C^p_{s} \arrow[u, dashed, shift left, "\text{E metr. and barr.}"] \arrow[r] & (G_{s})
\end{tikzcd}$$

Applications

- Mathematical physics
- Infinite dimensional Lie groups
- Gauge groups
- Manifolds of maps
- Diffeomorphism groups

Mathematical physics: phase spaces of PDEs are Fréchet spaces

Ex: 1) Electrodynamics : Maxwell's equations as Hamiltonian system on space of smooth vector potentials $\mathcal{V}ec$ Fréchet space

2) Ricci flow $g'(t) = -2Ric(g(t))$ (R. Hamilton, G. Perelman, Poincaré Conjecture)
Dynamical system (vector field) on space $\mathcal{M}$ of Riemannian metrics on M
$\mathcal{M}$ is a Fréchet manifold

3) Einstein's field equations $Ric(g)=0$ are invariant under $Diff^{\infty}(M)$ (coord.transf.)
Hamiltonian system on $\mathcal{M}/Diff^{\infty}(M)$ Fréchet manifold

4) **Quantum field theories**, QED, QCD, gauge theories

configuration spaces: spaces of vector potentials, Yang Mills fields are Fréchet spaces

gauge groups $\mathcal{G}$ (symmetries) are Fréchet manifold (Lie groups)

5) **Fluid dynamics:**

a) $$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p \quad , \quad div\ u = 0$$

Euler's equations of incompressible fluids are geodesics on the space of volume preserving diffeomorphisms

$Diff^{\infty}_{vol}(M)$ Fréchet manifold

b) The **KdV** eqaution $u_t + 6uu_x + u_{xxx} = 0$ for shallow water waves are a Hamiltonian system on the space of invertible Fourier integral operators $\mathcal{FIO}$ Fréchet manifold

6) **Plasma physics**: Maxwell-Vlasov equations are a Hamiltonian system on

$$\mathcal{MV} = (T^{*}Diff^{\infty}_{can}(\mathbf{R}^6) \times T^{*}\mathcal{V}ec)/C^{\infty}(\mathbf{R}^6)$$ Fréchet manifold

Manifolds of maps
Diffeomorphism groups

$C^\infty(M,N) = \{ f:M \to N \mid \text{smooth}, C^\infty \}$ **Fréchet manifold**

$\mathrm{Diff}^\infty(M) = \{ f:M \to M \mid \text{smooth}, C^\infty \text{ diffeomorphism} \}$ **Fréchet Lie group**

What does $C^\infty(M,N)$, $\mathrm{Diff}^\infty(M)$ locally look like ?

Fréchet space

i.e. parameter space E_f at $f \in C^\infty(M,N)$?

$E_f = T_f C^\infty(M,N) = \{ X_f: M \to TN \mid C^\infty \text{ vector field along } f \} = \Gamma^\infty(\pi_f)$

smooth C^∞ sections $\Gamma^\infty(\pi_f)$ of vector bundle π_f : **Fréchet space**

$C^\infty(M,N)$, $\mathrm{Diff}^\infty(M)$ smooth, C^∞_Γ manifolds

composition $c: \mathrm{Diff}^\infty(M) \times \mathrm{Diff}^\infty(M) \to \mathrm{Diff}^\infty(M)$: $c(g,f) = g \circ f$ is C^∞_Γ

inversion $i: \mathrm{Diff}^\infty(M) \to \mathrm{Diff}^\infty(M)$: $i(f) = f^{-1}$ is C^∞_Γ

$\mathrm{Diff}^\infty(M)$ is a C^∞_Γ Fréchet Lie group

Other approach: **I**nverse**L**imit**H**ilbert **= ILH** **,** **I**nverse**L**imit**B**anach **= ILB**

$\mathrm{Diff}^{\infty}(M) = \underset{\leftarrow}{\lim}\mathrm{Diff}^{s}(M)$ H^{s} Sobolev class **ILH Lie group**

$\mathrm{Diff}^{\infty}(M) = \underset{\leftarrow}{\lim}\mathrm{Diff}^{k}(M)$ C^{k} class **ILB Lie group**

Pseudo differential operators $\Psi\mathcal{DO}$ **ILH Lie group**

Fourier integral operators $\mathcal{FIO}$ **ILH Lie group**