

THE GEOMETRY OF EULER'S EQUATIONS

TOPOLOGICAL EULER'S EQUATIONS AND DIFFEOMORPHISM GROUPS

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1. Classical Euler Equations

$$(E_{cl}) \quad \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) & = \text{grad } p \\ \text{div } u(t) & = 0 \end{cases}$$

Let (M, g) be a smooth **compact** Riemannian manifold with volume μ and $\text{Diff}_\mu^s(M) = \{f : M \rightarrow M \mid \text{volume preserving } H^s\text{-diffeomorphism}, f^*\mu = \mu\}$ the volume preserving diffeomorphisms

Theorem: (Ebin-Marsden, 1970)

$\text{Diff}_\mu^s(M)$ is an infinite dimensional Riemannian manifold (Lie group) and the Euler equations (E_{cl}) are equivalent to geodesics on $\text{Diff}_\mu^s(M)$.

- short time existence of unique solutions
- smooth regularity of solutions

Notice:

- 1) M compact
- 2) $\mu =$ Riemannian volume form of metric g
- 3) $\nabla = \nabla^g$ the covariant derivative of the metric g .

We generalize this theorem to open, **non compact** manifolds and to μ **any volume form** on M (not Riemannian).

2. NON-COMPACT case: (Eichhorn, Schmid)

Eichhorn:

"In the non-compact case there is exactly one thing that works like in the compact case: nothing !"

Idea: Bounded Geometry

- Control over the metric and its derivatives
- Control over the mappings and their derivatives by the metric, i.e. maps adapted to the geometry

Definition: A Riemannian manifold (M^n, g) has **bounded geometry of order k** , $0 \leq k \leq \infty$, if M has a positive injectivity radius $r_{inj}(M)$ and the curvature tensor R and all its derivatives up to order k are uniformly bounded; i.e the following two conditions (I) and (B_k) are satisfied:

$$(I) : r_{inj}(M) = \inf_{x \in M} r_{inj}(x) > 0$$

$$(B_k) : |\nabla^i R| \leq C_i, \quad 0 \leq i \leq k.$$

Examples of manifolds with bounded geometry:

- compact manifolds
- Lie groups
- homogeneous spaces
- covering spaces of Riemannian manifolds
- leaves of foliations of compact manifolds

These conditions I and B_k can be expressed as follows:

(I) \Leftrightarrow there exists a ball around 0 in \mathbf{R}^n which is domain of normal (geodesic) coordinates **for all** $x \in M$.

(B_k) \Leftrightarrow there exists a constant d_k (independent of $x \in M$) such that

$\|g_{ij}\|_{C^k} \leq d_k$ in any normal coordinate system

$\Leftrightarrow |D^\alpha g_{ij}| \leq c_\alpha, |\alpha| \leq k$ in any normal coordinate system

$\Leftrightarrow \|\Gamma_{ij}^m\|_{C^{k-1}} \leq d'_k$ in any normal coordinate system

$\Leftrightarrow |D^\beta \Gamma_{ij}^m| \leq c_\beta, |\beta| \leq k - 1$, in any normal coordinate system.

Fact: There is **no** topological obstruction for a metric with bounded geometry of any order.

Bounded maps $C^{\infty,m}(M, N)$:

Consider now $(M^n, g), (N^{n'}, h)$ open, complete Riemannian manifolds satisfying (I) and (B_k) and $f \in C^\infty(M, N)$. Then the differential $f_* = Tf$ is a section of $T^*M \otimes f^*TN$. We endow f^*TN with the induced connection $f^*\nabla^h$. Then ∇^g and $f^*\nabla^h$ induce connections ∇ in all tensor bundles $T_s^q(M) \otimes f^*T_v^u(N)$. Therefore $\nabla^m df$ is well defined. Assume $m \leq k$. We denote by $C^{\infty,m}(M, N)$ the set of all $f \in C^\infty(M, N)$ satisfying

$${}_{b,m} |df| := \sum_{i=0}^{m-1} \sup_{x \in M} |\nabla^i df|_x < \infty.$$

Equivalently:

$f \in C^{\infty,m}(M, N) \Leftrightarrow \frac{\partial^\alpha}{\partial x^\alpha} f^\nu$ is uniformly bounded in any normal coordinate system; $|\alpha| \leq m, 1 \leq \nu \leq n'$.

What is the topology and geometry of $C^{\infty,m}(M, N)$?

The bounded diffeomorphism group $Diff^{p,m}(M)$.

Problem: $C^{\infty,m}(M) \cap Diff(M)$ is **not** a group, i.e
 $f \in C^{\infty,m}(M) \cap Diff(M) \not\Rightarrow f^{-1} \in C^{\infty,m}(M)$.

We need an additional assumption :

Let (M^n, g) be an oriented, open, complete Riemannian manifold of bounded geometry of order k , $1 < p < \infty$, $k \geq m > \frac{n}{p} + 1$. A choice of an orthonormal basis in each $T_x M$ implies that $|\lambda|_{min}(df)$, the absolute value of the eigenvalues of the Jacobian of f , is well defined. Set

$Diff^{p,m}(M) := \{f \in \Omega^{p,m}(M, M) \mid f \text{ orientation pres. isom. } |\lambda|_{min}(df) > 0\}$.

Theorem: Each component of $Diff^{p,m}$ is a C^{k+1-m} Banach manifold, and for $p = 2$ it is a Hilbert manifold.

Theorem: Assume $(M^n, g), k, p, m$ as above.

a) Assume $f, g \in Diff^{p,m}, g \in comp(id_M) \subset Diff^{p,m}$. Then $g \circ f \in Diff^{p,m}$ and $g \circ f \in comp(f)$.

b) Assume $f \in comp(id_M) \subset Diff^{p,m}$.

Then $f^{-1} \in comp(id) \subset Diff^{p,m}$.

c) $comp(id)$ is a metrizable topological group.

Volume preserving and symplectic diffeomorphisms

Theorem: Assume (M^n, g) is an open manifold satisfying the conditions (I) and $(B_k), k \geq m \geq r > \frac{n}{2} + 1$ and the spectral condition $\inf \sigma_{ess}(\Delta_1|_{(ker \Delta_1)^\perp}) > 0$. Let ω be a C^m -bounded closed q -form with $\inf_{x \in M} |\omega|_x^2 > 0$, and consider $Diff_\omega^r = \{f \in Diff^r | f^*\omega = \omega\}$.

Then the group $Diff_{\omega,o}^r = Diff_o^r \cap Diff_\omega^r$ is a C^{k-r+1} submanifold of $Diff_o^r$.

Theorem: Assume (M^n, g) with (I) and (B_∞) . Let ω be a C^∞ -bounded strongly nondegenerate closed q -form, $q = n$ or $q = 2$, and assume the spectral condition above. Set $Diff_{\omega,o}^\infty = \lim_{\leftarrow r} Diff_{\omega,o}^r$.

Then $\{Diff_{\omega,o}^\infty, Diff_{\omega,o}^r | r > \frac{n}{2} + 1\}$ is an ILH-Lie group and the Lie algebra of $Diff_{\omega,o}^\infty$ consists of divergence free ($q = n$), or locally Hamiltonian ($q = 2$) vector fields X with finite Sobolev norm $|X|_r$ for all r .

Applications

1) **Classical Euler equations** for an incompressible, homogeneous fluid without viscosity

$$E_{cl} \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) = \text{grad } p \\ \text{div } u(t) = 0 \end{cases}$$

where $u = u(x, t)$ is a time dependent C^1 vector field on (M^n, g) , $\nabla = \nabla^g$, $\text{div} = \text{div}_{dvol_x(g)}$. Additionally, we assume $u(t) \in \Omega^r(TM)$ for all t which means that the fluid moves very slowly at infinity, $r > \frac{n}{2} + 1$. Then $u(t)$ defines a 1-parameter family of diffeomorphisms f_t defined by

$$\left. \frac{df_s}{ds} \right|_{s=t} = u(t) \circ f_t .$$

The f_t remain in the identity component of $\text{Diff}_{\mu}^{\infty, r}(M)$, since $f_0 = id$, $\text{div } u = 0$, and $\mu = dvol_x(g)$.

Theorem: Assume (M^n, g) with (I) and (B_k) , $\inf \sigma_e(\Delta_1|_{(\ker \Delta_1)^\perp}) > 0$, $k - 2 \geq r > \frac{n}{2} + 1$. Then $u(t)$ satisfies the classical Euler equations (E_{cl}) iff $\{f_t\}_t$ is a geodesic in $\text{Diff}_{\mu, 0}^{\infty, r}(M)$.

- short time existence of unique solutions
- smooth regularity of solutions

2) Topological Euler equations

μ = fixed volume form on (M, g)

$u = u(x, t)$ is a time dependent C^1 vector field on (M^n, g)

$\nabla = \nabla^g$ the Riemannian covariant derivative

but now **div** = **div** $_{\mu}$, defined by $L_X \mu = (\text{div}_{\mu} X) \mu$.

The **topological Euler equations** are given by

$$(E_{top}) \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) = \text{grad } p \\ \text{div}_{\mu} u(t) = 0 \end{cases}$$

Theorem: Assume (M^n, g) with (I) and (B_k) , $\inf \sigma_e(\Delta_1|_{(\ker \Delta_1)^{\perp}}) > 0$, $k - 2 \geq r > \frac{n}{2} + 1$.

Then $u(t)$ satisfies the topological Euler equations (E_{top}) iff $\{f_t\}_t$ is a geodesic in $\text{Diff}_{\mu, 0}^{\infty, r}(M)$.

- short time existence of unique solutions
- smooth regularity of solutions

3) Non-homogeneous Euler equations

The **non-homogeneous Euler equations** with a mass density $\rho(x, t) > 0$ are given by

$$E_{NH} \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) = \frac{1}{\rho} \text{grad } p \\ \frac{\partial \rho}{\partial t} + (\text{grad } \rho) \cdot u = 0 \\ \text{div}_{\mu} u(t) = 0 . \end{cases} \quad (1)$$

If $\rho = \text{constant}$ these are the classical homogeneous Euler equations. For $\mu = \mu(g)$ the corresponding equations on $\text{Diff}_{\mu}^s(M)$ are **not** right invariant, i.e. they are not derivable from Arnold's method as above. But if we take $\tilde{\mu} = \rho_o \mu(g)$ as volume form, then we have the following

Theorem: (Eichhorn-Schmid)

$u(t)$ is a solution of $E_{NH} \Leftrightarrow (f_t)_t$ the flow of $u(t)$ is a geodesic on $\text{Diff}_{\tilde{\mu}}^s(M)$ and $\rho(x, t) = \rho_o(f_t^{-1}(x))$, where the volume form is $\tilde{\mu} = \rho_o \mu(g)$.

work in progress