TOPOLOGICAL EULER EQUATIONS AND DIFFEOMORPHISM GROUPS

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1. Classical Euler Equations

$$(E_{cl})\ \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) &= grad\ p \\ div\ u(t) &= 0 \end{cases}$$

Let (M, g) be a smooth **compact** Riemannian manifold with volume μ and $Diff^s_\mu(M) = \{f : M \to M | volume\ preserving H^s - diffeomorphism,\ \ f^*\mu = \mu\}$ the volume preserving diffeomorphisms

Theorem: (Ebin-Marsden, 1970)
$Diff^s_\mu(M)$ is an infinite dimensional Riemannian manifold (Lie group) and the Euler equations (E_{cl}) are equivalent to geodesics on $Diff^s_\mu(M)$.

- **short time existence of unique solutions**
- **smooth regularity of solutions**

Notice:
1) M compact
2) μ = Riemannian volume form of metric g
3) $\nabla = \nabla^g$ the covariant derivative of the metric g.

We generalize this theorem to open , **non compact** manifolds and and to μ **any volume form** on M (not Riemannian).

2. NON-COMPACT case: (Eichhorn, Schmid)

Eichhorn:
"In the non-compact case there is exactly one thing that works like in the compact case: nothing ! "

Idea: Bounded Geometry

- Control over the metric and its derivatives
- Control over the mappings and their derivatives by the metric,

i.e. maps adapted to the geometry

Definition: A Riemannian manifold (M^n, g) has **bounded geometry of order** k, $0 \leq k \leq \infty$, if M has a positive injectivity radius $r_{inj}(M)$ and the curvature tensor R and all is derivatives up to order k are uniformly bounded; i.e the following two conditions (I) and (B_k) are satisfied:

$$(I): \; r_{inj}(M) = \inf_{x \in M} r_{inj}(x) > 0$$

$$(B_k): \; |\nabla^i R| \leq C_i \;,\; 0 \leq i \leq k.$$

Examples of manifolds with bounded geometry:

- compact manifolds
- Lie groups
- homogeneous spaces
- covering spaces of Riemannian manifolds
- leaves of foliations of compact manifolds

These conditions I and B_k can be expressed as follows:

(I) $\Leftrightarrow$ there exists a ball around 0 in $\mathbf{R}^n$ which is domain of normal (geodesic) coordinates **for all** $x \in M$.

$(B_k) \Leftrightarrow$ there exists a constant d_k (independent of $x \in M$) such that
$\|g_{ij}\|_{C^k} \leq d_k$ in any normal coordinate system
$\Leftrightarrow |D^\alpha g_{ij}| \leq c_\alpha$, $|\alpha| \leq k$ in any normal coordinate system
$\Leftrightarrow \|\Gamma^m_{ij}\|_{C^{k-1}} \leq d'_k$ in any normal coordinate system
$\Leftrightarrow |D^\beta \Gamma^m_{ij}| \leq c_\beta$, $|\beta| \leq k-1$, in any normal coordinate system.

Fact: There is **no** topological obstruction for a metric with bounded geometry of any order.

Bounded maps $C^{\infty,m}(M,N)$:

Consider now $(M^n,g),(N^{n'},h)$ open, complete Riemannian manifolds satisfying (I) and (B_k) and $f \in C^\infty(M,N)$. Then the differential $f_* = Tf$ is a section of $T^*M \otimes f^*TN$. We endow f^*TN with the induced connection $f^*\nabla^h$. Then ∇^g and $f^*\nabla^h$ induce connections ∇ in all tensor bundles $T^q_s(M) \otimes f^*T^u_v(N)$. Therefore $\nabla^m df$ is well defined. Assume $m \le k$. We denote by $C^{\infty,m}(M,N)$ the set of all $f \in C^\infty(M,N)$ satisfying

$$^{b,m}|df| := \sum_{i=0}^{m-1} \sup_{x\in M} |\nabla^i df|_x < \infty.$$

Equivalently:
$f \in C^{\infty,m}(M,N) \Leftrightarrow \frac{\partial^\alpha}{\partial x^\alpha} f^\nu$ is uniformly bounded in any normal coordinate system; $|\alpha| \le m, 1 \le m \le k$.

What is the topology and geometry of $C^{\infty,m}(M,N)$?

The bounded diffeomorphism group $Diff^{p,m}(M)$.

Problem: $C^{\infty,m}(M) \bigcap Diff(M)$ is **not** a group, i.e
$f \in C^{\infty,m}(M) \bigcap Diff(M) \not\Rightarrow f^{-1} \in C^{\infty,m}(M)$.
We need an additional assumption :
Let (M^n, g) be an oriented, open, complete Riemannian manifold of bounded geometry of order k, $1 < p < \infty$, $k \geq m > \frac{n}{p} + 1$. A choice of an orthonormal basis in each T_xM implies that $|\lambda|_{min}(df)$, the absolute value of the eigenvalues of the Jacobian of f, is well defined. Set

$$Diff^{p,m}(M) := \{f \in \Omega^{p,m}(M, M) \mid f \ orientation \ pres. \ isom. \ |\lambda|_{min}(df) > 0\}.$$

Theorem: Each component of $Diff^{p,m}$ is a C^{k+1-m} Banach manifold, and for $p = 2$ it is a Hilbert manifold.

Theorem: Assume $(M^n, g), k, p, m$ as above.
a) Assume $f, g \in Diff^{p,m}, g \in \ comp\ (id_M) \subset Diff^{p,m}$. Then $g \circ f \in Diff^{p,m}$ and $g \circ f \in \ comp\ (f)$.
b) Assume $f \in \ comp(id_M) \subset Diff^{p,m}$.
Then $f^{-1} \in \ comp(id) \subset Diff^{p,m}$.
c) $comp(id)$ is a metrizable topological group.

Volume preserving and symplectic diffeomorphisms

Theorem: Assume (M^n, g) is an open manifold satisfying the conditions (I) and $(B_k), k \geq m \geq r > \frac{n}{2}+1$ and the spectral condition inf $\sigma_{ess}(\Delta_1|_{(ker\Delta_1)^\perp}) > 0$. Let ω be a C^m-bounded closed q-form with $\inf_{x\in M} |\omega|_x^2 > 0$, and consider $Diff^r_\omega = \{f \in Diff^r | f^*\omega = \omega)\}$.
Then the group $Diff^r_{\omega,o} = Diff^r_o \cap Diff^r_\omega$ is a C^{k-r+1} submanifold of $Diff^r_o$.

Theorem: Assume (M^n, g) with (I) and (B_∞) . Let ω be a C^∞-bounded strongly nondegenerate closed q-form, $q = n$ or $q = 2$, and assume the spectral condition above. Set $Diff^\infty_{\omega,o} = \lim_{\leftarrow r} Diff^r_{\omega,o}$.
Then $\{Diff^\infty_{\omega,o}, Diff^r_{\omega,o}| \, r > \frac{n}{2}+1\}$ is an ILH-Lie group and the Lie algebra of $Diff^\infty_{\omega,o}$ consists of divergence free ($q = n$), or locally Hamiltonian $(q = 2)$ vector fields X with finite Sobolev norm $|X|_r$ for all r.

Applications

1) **Classical Euler equations** for an incompressible, homogeneous fluid without viscosity

$$E_{cl} \left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) & = & grad\ p \\ div\ u(t) & = & 0 \end{array} \right.$$

where $u = u(x,t)$ is a time dependent C^1 vector field on (M^n, g), $\nabla = \nabla^g$, $div = div_{dvol_x(g)}$. Additionally, we assume $u(t) \in \Omega^r(TM)$ for all t which means that the fluid moves very slowly at infinity , $r > \frac{n}{2} + 1$. Then $u(t)$ defines a 1-parameter family of diffeomorphisms f_t defined by

$$\frac{df_s}{ds}|_{s=t} \;=\; u(t) \circ f_t \ .$$

The f_t remain in the indentity component of $Diff_\mu^{\infty,r}(M)$, since $f_0 = id$, $div\ u = 0$, and $\mu = dvol_x(g)$.

Theorem: Assume (M^n, g) with (I) and (B_k), $\inf \sigma_e(\Delta_1|_{(ker\ \Delta_1)^\perp}) > 0$, $k - 2 \geq r > \frac{n}{2} + 1$. Then $u(t)$ satisfies the classical Euler equations (E_{cl}) iff $\{f_t\}_t$ is a geodesic in $Diff_{\mu,0}^{\infty,r}(M)$.

- **short time existence of unique solutions**
- **smooth regularity of solutions**

2) Topological Euler equations

$\mu =$ fixed volume form on (M, g)
$u = u(x,t)$ is a time dependent C^1 vector field on (M^n, g)
$\nabla = \nabla^g$ the Riemannian covariant derivative
but now $\mathbf{div} = \mathbf{div}_\mu$, defined by $L_X\mu = (div_\mu X)\mu$.

The **topological Euler equations** are given by

$$(E_{top}) \left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} + \nabla_{u(t)}u(t) & = & grad\ p \\ div_\mu\, u(t) & = & 0 \end{array} \right.$$

Theorem: Assume (M^n, g) with (I) and (B_k), $\inf \sigma_e(\Delta_1|_{(ker\ \Delta_1)^\perp}) > 0$, $k - 2 \geq r > \frac{n}{2} + 1$.
Then $u(t)$ satisfies the topological Euler equations (E_{top}) iff $\{f_t\}_t$ is a geodesic in $Diff_{\mu,0}^{\infty,r}(M)$.

- **short time existence of unique solutions**
- **smooth regularity of solutions**

3) Non-homogeneous Euler equations

The **non-homogeneous Euler equations** with a mass density $\rho(x,t) > 0$ are given by

$$E_{NH} \left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) & = & \frac{1}{\rho} \text{ grad } p \\ \frac{\partial \rho}{\partial t} + (\text{grad } \rho) \cdot u & = & 0 \\ \text{div}_\mu \, u(t) & = & 0 \; . \end{array} \right. \tag{1}$$

If $\rho =$ constant these are the classical homogeneous Euler equations. For $\mu = \mu(g)$ the corresponding equations on $Diff^s_\mu(M)$ are **not** right invariant, i.e. they are not derivable from Arnold's method as above. But if we take $\tilde{\mu} = \rho_o \mu(g)$ as volume form, then we have the following

Theorem: (Eichhorn-Schmid)
$u(t)$ is a solution of $E_{NH} \Leftrightarrow (f_t)_t$ the flow of $u(t)$ is a geodesic on $Diff^s_{\tilde{\mu}}(M)$ and $\rho(x,t) = \rho_o(f_t^{-1}(x))$, where the volume form is $\tilde{\mu} = \rho_o \mu(g)$.

work in progress