THE LIE GROUP OF FOURIER INTEGRAL OPERATORS * AND APPLICATIONS TO FLUID DYNAMICS

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* *The Lie Group of Fourier Integral Operators on* **open** *Manifolds*,

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Review COMPACT case:

Theorem: (M.Adams, T.Ratiu, R.Schmid, 1985) :
The group $FIO_*(M)$ of invertible Fourier integral operators on a compact manifold M is a graded ∞-dim Lie group with graded ∞-dim Lie algebra $\Psi DO(M)$ of pseudodifferential operators on M.

$FIO_*(M)$ is and ∞-dim principal fiber bundle over the base manifold $Diff_\theta(\dot{T}^*M)$ of contact transformations of $\dot{T}^*M$ with gauge group $\Psi DO_*(M)$ of invertible pseudodifferential operators.

$$\begin{array}{ccc} \Psi DO_*(M) & \longrightarrow & FIO_*(M) \\ & & \downarrow \\ & & Diff_\theta(\dot{T}^*M) \end{array}$$

- FIO, Fourier integral operators, they generalize
- ΨDO, pseudodifferential operators, they generalize
- DO, differential operators

$$FIO \supset \Psi DO \supset DO$$

$P \in \Psi DO_m$ classical ΨDO order m ,locally : $u \in C_c^\infty(M)$

$$Pu(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} p(x,\xi)u(y)dy\ d\xi$$

- $p(x,\xi)$ classical symbol of order m

$$p(x,\xi) \sim \sum_{j=m}^{-\infty} p_j(x,\xi)\ ,\quad \textit{homogeneous degree } j \textit{ in } \xi$$

- special case:

$$p(x,\xi) = \sum_{|\alpha|\le m} p_\alpha(x)\xi^\alpha \quad \textit{polynomial in } \xi$$

$\Rightarrow\ P \in DO$ differential operator order m

P oscillatory integral, highly singular , nice operators:
(i) invariant under diffeomorphisms, $\Rightarrow$ def. on M
(ii) extend $P : H_c^s(M) \to H_c^{s-m}(M)$ bounded
(iii) closed under composition, order additive
(iv) $\Psi DO = \bigcup_m \Psi DO_m$ is an ∞-dim graded Lie algebra

$$P \in \Psi DO_m\ ,\ Q \in \Psi DO_n$$
$$\Rightarrow [P,Q] = PQ - QP \in \Psi DO_{m+n-1}$$

Note: ΨDO_1 is ∞-dim Lie algebra : ?? LIE GROUP ??

Question: Are there ∞-dim Lie groups $\mathcal{G}$, $\mathcal{G}_o$ which have ΨDO resp. ΨDO_1 as Lie algebras?
Answer: YES $\mathcal{G} = FIO_*$ and $\mathcal{G}_o = (FIO_o)_*$ the groups of invertible Fourier integral operators (order zero)

$A \in FIO_m$ Fourier integral operator of order m on M, compact, locally:

$$Au(x) = (2\pi)^{-n} \int \int e^{i\varphi(x,y,\xi)} a(x,\xi) u(y) dy\ d\xi$$

- $a(x,\xi)$ classical symbol of order m , $\quad a \sim \sum_{j=m}^{-\infty} a_j$
- $\varphi(x,y,\xi)$ nondegenerate phase function

Note: if $\varphi(x,y,\xi) = (x-y)\cdot\xi \Rightarrow A \in \Psi DO_m$

φ is locally generating a homogeneous canonical transformation $\eta : \dot{T}^*M \to \dot{T}^*M : \eta^*\omega = \omega, \eta(t\alpha) = t\eta(\alpha)$
$\Leftrightarrow$ $\eta^*\theta = \theta$ canonical 1-form (quantomorphism)

again
(i) $A : C_c^\infty(M) \to C^\infty(M)$ extends $A : \mathcal{E}'(M) \to \mathcal{D}'(\mathcal{M})$
(ii) closed under composition, order additive, and if $A = A_1 \circ A_2$ then $\eta = \eta_1 \circ \eta_2$, and $A^{-1} \Rightarrow \eta^{-1}$

Let
$FIO = \bigcup_m FIO_m$, FIO_* group of invertible $FIOs$
$Diff_\theta^\infty = \{\eta \in Diff^\infty(\dot{T}^*M) \mid \eta^*\theta = \theta\}$

we have an exact sequence of groups

$$I \to \Psi DO_* \overset{j}{\hookrightarrow} FIO_* \overset{p}{\to} Diff^\infty_\theta \to id$$

$A \in FIO_*$, $p(A) = \eta \in Diff^\infty_\theta$ surjective,
$ker\ p = \Psi DO_*$, since $\varphi(x,y,\xi) = (x-y)\cdot\xi \Rightarrow id$

we want : LIE GROUPS

zero order groups

$$I \to (\Psi DO_0)_* \overset{j}{\hookrightarrow} (FIO_0)_* \overset{p}{\to} Diff^\infty_\theta \to id$$

Lie algebras

$$0 \longrightarrow \Psi DO_0 \hookrightarrow \Psi DO_1 \longrightarrow Vec^\infty_\theta \longrightarrow 0$$

$$Vec^\infty_\theta \ \| \ L_X\theta = 0$$

Note !!

$$Diff^\infty_\theta \neq exp(Vec^\infty_\theta) \quad , \quad FIO \neq exp(\Psi DO)$$

Idea: construct ∞-dim principal fiber bundle such that

- base space $= Diff^\infty_\theta(\dot{T}^*M)$
- total space $= (FIO_0)_*$
- fiber $= p^{-1}(\eta) \simeq (\Psi DO_0)_* =$ gauge group

step 1: $Diff_\theta^\infty = \lim_{\infty \leftarrow s} Diff_\theta^s$ is an ILH Lie group

step 2: $(\Psi DO_0)_* = \lim_{\infty \leftarrow s} (\Psi DO_0^s)_*$ is an ILH Lie group

step 3: piece 1 & 2 together via local section σ

$$\sigma : U \subset Diff_\theta^\infty \to (FIO_0)_*$$

then $(FIO_0)_*$ locally: $p^{-1}(U) \simeq \sigma(U) \times (\Psi DO_0)_*$
$\Rightarrow$ chart at identity $I \in (FIO_0)_*$

step 4: move this chart around by group structure of $Diff_\theta^\infty$
step 5: chart transitions smooth $\Rightarrow (FIO_0)_*$ smooth manifold
step 6: multiplication "smooth" $\Rightarrow (FIO_0)_*$ Lie group
step 7: identify $(1-\Delta)^{m/2} : (FIO_0)_* \xrightarrow{\simeq} (FIO_m)_*$

Theorem: (M. Adams, T. Ratiu, R. Schmid, 1985) : The group $FIO_*(M)$ of invertible Fourier integral operators on a compact manifold M is a graded ∞-dim ILH-Lie group with graded ∞-dim Lie algebra $\Psi DO(M)$ of pseudodifferential operators on M.
$FIO_*(M)$ is and ∞-dim principal fiber bundle over the base manifold $Diff_\theta^s(\dot{T}^*M)$ of contact transformations of $\dot{T}^*M$ with gauge group $\Psi DO_*(M)$ of invertible pseudodifferential operators.

Next: **NON-COMPACT** case

J. Eichhorn:
"There is exactly one thing that work in the non-compact case: NOTHING"

Diffeomorphisms of NON-COMPACT manifolds:

Example: Let M^n and $N^{n'}$ be compact manifolds, then a map $f : M^n \to N^{n'}$ is of Sobolev class H^s if and only if the local representatives $f^i_j : U_i \subset \mathbf{R}^n \to V_j \subset \mathbf{R}^{n'}$ are of class H^s, where $M \subset \bigcup(U_i, \phi_i),\ N \subset \bigcup(V_j, \psi_j),\ f^i_j := \psi_j \circ f \circ \phi_i^{-1}$.
These covers are **finite** if M, N are compact.
This definition is invariant $\Leftrightarrow\ s > \frac{n}{2} + 1$
In the compact case we can define the distance by

$$d^s(f,g) := (\sum_{i,j} \|f^i_j - g^i_j\|^2_s)^{\frac{1}{2}}$$

These definitions are meaningless if M, N are open !

Idea: Bounded Geometry

- Control over the metric and its derivatives
- Control over the mappings and their derivatives by the metric, i.e. maps adapted to the bounded geometry

Definition: A Riemannian manifold (M^n, g) has *bounded geometry of order* k, $0 \leq k \leq \infty$, if M has a positive injectivity radius $r_{inj}(M)$ and the curvature tensor R and all is derivatives up to order k are uniformly bounded; i.e the following two conditions (I) and (B_k) are satisfied:

$$(I): \ r_{inj}(M) = \inf_{x \in M} r_{inj}(x) > 0$$

$$(B_k): \ |\nabla^i R| \leq C_i \ , \ 0 \leq i \leq k.$$

$(I) \Leftrightarrow$ there exists a ball around 0 in $\mathbf{R}^n$ which is domain of normal (geodesic) coordinates **for all** $x \in M$.

$(B_k) \Leftrightarrow$ there exists a constant d_k (independent of $x \in M$) such that $\|g_{ij}\|_{C^k} \leq d_k$ in any normal coordinate system $\Leftrightarrow \|\Gamma^m_{ij}\|_{C^{k-1}} \leq d_k$ in any normal coordinate system

Examples of manifolds with bounded geometry:
- compact manifolds
- Lie groups
- homogeneous spaces
- covering spaces of Riemannian manifolds
- leaves of foliations of compact manifolds

Fact: Given an open manifold M^n and $k \geq 0$, then there exists a complete Riemannian metric g on M^n satisfying the conditions (I) and (B_k); i.e there is **no** topological obstruction for a metric with bounded geometry of any order.

Bounded maps $C^{p,r}(M,N)$:
A C^∞ map $f : M \to N$ is bounded of order $r \leq k$ iff $\frac{\partial^\alpha}{\partial x^\alpha} f^\nu$ is uniformly bounded in any normal coordinate system; $|\alpha| \leq r \leq k$.

Topology of bounded maps: f, g are close iff ex. a vector field ξ along f with small Sobolev norm $\|\xi\|_{p,r} < \varepsilon$ such that $g(x) = exp_{f(x)}\xi(x)$

Theorem: (J. Eichhorn, R. Schmid)
The completion $C^{p,r}(M,N)$ is a C^{k+1-r}-Banach manifold, and for $p = 2$ it is a Hilbert manifold
$1 < p < \infty$,$r \leq k$, $r > \frac{n}{p} + 1$

The bounded diffeomorphism group $Diff^{p,r}(M)$ **.**

Problem: f bounded $\not\Rightarrow f^{-1}$ bounded , i.e no goup .
We need an additional assumption then $Diff^{p,r}$ is open in $C^{p,r}(M,M)$ and each component is a C^{k+1-r}-Banach manifold, and for $p = 2$ it is a Hilbert manifold.

Theorem: (J. Eichhorn, R. Schmid) Let (M^n, g) be an open, oriented, complete Riemannian manifold satisfying (I), (B_∞) and let $r > \frac{n}{p} + 1$. Then $Diff^{p,\infty}(M) = \lim_{\leftarrow} Diff^{p,r}(M)$ is an ILB - Lie group; and for $p = 2$ it is an ILH - Lie group.

Volume preserving and symplectic diffeomorphisms

Let ω be a C^∞-bounded non-degenerate q-form, $q = n$ or $q = 2$, consider $Diff_\omega^{p,r} = \{f \in Diff^r | f^*\omega = \omega\}$.

Theorem: (Eichhorn , Schmid)
a) $Diff_\omega^{p,\infty} = \lim_{\leftarrow r} Diff_\omega^{p,r}$ is an ILH-Lie group with Lie algebra consisting of divergence free ($q = n$), or locally Hamiltonian ($q = 2$) vector fields ξ with finite Sobolev norm $|\xi|_{p,r}$ for all r.
b) $Diff_\omega^{p,r}$ is an infinite dim. Riemannian manifold, with (weak) metric

$$g(X,Y)_{id} = \int_M (X,Y)_x dvol_x(g)$$

Contact transformations on $\dot{T}^*M$

If (M^n, g) is an open, oriented, complete Riemannian manifold satisfying (I), (B_k) then the Sasaki metric on the co-sphere bundle in $\dot{T}^*M$ satisfies (I), (B_{k-1})
Let θ be the canonical 1-form on T^*M and consider

$$Diff_\theta^{p,r} = \{f \in Diff^{p,r}(\dot{T}^*M) |\ f^*\theta = \theta\ \}$$

. **Theorem:** (Eichhorn, Schmid)
$Diff_\theta^{p,\infty} = \lim_{\leftarrow r} Diff_\theta^{p,r}$ is an ILH-Lie group.

This is the space of phace functions for the Fourier integral operators !

Pseudodifferential operators and Fourier integral operators on open manifolds

If (M^n, g) is **open** the previous definition of ΨDOs and $FIOs$ does **not** make sense. We need to adapt the class of symbols and phase functions to the bounded geometry of M in order to obtain globally defined Fourier integral operators $A : C_c^\infty(M) \to \mathcal{D}'(M)$. Then the corresponding spaces ΨDO and FIO have similar properties as in the compact case and we can use the same ideas as before to construct Lie group structures.

$$FIO \qquad Au(x) = (2\pi)^{-n} \int\int e^{i\varphi(x,y,\xi)} a(x,\xi) u(y) dy\ d\xi$$

- **symbols:** the family of local symbols $a(x,\xi)$ together with their derivatives should be uniformly bounded

- **phase functions:** the phase functions $\varphi(x,y,\xi)$ should locally generate canonical transformations in the space $Diff_\theta^{p,r}(\dot{T}^*M)$

This defines the class of **uniform** pseudodifferential- and Fourier integral operators $\quad \mathcal{U}\Psi DO_m, \quad \mathcal{U}FIO_m$

Again we get an exact sequence of groups

$$I \to (\mathcal{U}\Psi DO_0)_* \overset{j}{\hookrightarrow} (\mathcal{U}FIO_0)_* \overset{p}{\to} Diff_\theta^{p,r} \to id$$

Now we follow the same ideas as in the compact case: step 1,2...7 to construct ILH Lie groups structures on these spaces.

Main Theorem: (Eichhorn, Schmid)

$$\mathcal{U}\Psi DO = \lim_{\leftarrow s} \mathcal{U}\Psi DO^s \quad \text{is an ILH Lie group}$$
$$\mathcal{U}FIO = \lim_{\leftarrow t} \mathcal{U}FIO^t \quad \text{is an ILH Lie group}$$

multiplication
$\mu : \mathcal{U}FIO^{t+r} \times \mathcal{U}FIO^t \to \mathcal{U}FIO^t$; $\mu(A,B) = AB$

$$is\ C^k\ differentiable, k = min(r,t)$$

inversion $\nu : \mathcal{U}FIO^{t+r} \to \mathcal{U}FIO^t$; $\nu(A) = A^{-1}$

$$is\ C^k\ differentiable, k = min(r,t)$$

right mult. $R_A : \mathcal{U}FIO^t \to \mathcal{U}FIO^t$; $R_A(B) = BA$

$$is\ C^t\ differentiable, for\ any\ A \in \mathcal{U}FIO^t$$

left mult. $L_A : \mathcal{U}FIO^t \to \mathcal{U}FIO^t$; $L_A(B) = AB$

$$is\ C^0\ (continuous), for\ any\ A \in \mathcal{U}FIO^t$$

APPLICATIONS

1. Euler equations

1) Classical Euler equations for an incompressible, homogeneous fluid without viscosity

$$E_{cl}\ \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) &= grad\ p \\ div\, u(t) &= 0 \end{cases}$$

where $u = u(x,t)$ is a time dependent C^1 vector field on $(M^n, g), \nabla = \nabla^g,\ div = div_{dvol_x(g)}$. Additionally, we assume $u(t) \in \Omega^r(TM)$ for all t which means that the fluid moves very slowly at infinity , $r > \frac{n}{2} + 1$. Then $u(t)$ defines a 1-parameter family of diffeomorphisms f_t defined by

$$\frac{df_s}{ds}|_{s=t} = u(t) \circ f_t\ .$$

Theorem: (Eichhorn, Schmid) Assume (M^n, g) **open** with (I) and (B_k), $k-2 \geq r > \frac{n}{2} + 1$. Then $u(t)$ satisfies the classical Euler equations (E_{cl}) iff $\{f_t\}_t$ is a geodesic in $Diff_\mu^{\infty,r}(M)$.

M **compact**: D.G. Ebin and J.E. Marsden (1970).

2) Topological Euler equations

$\mu =$ fixed volume form on (M, g)
$u = u(x,t)$ is a time dependent C^1 vector field on (M^n, g)
$\nabla = \nabla^g$ the Riemannian covariant derivative
but now $div = div_\mu$, defined by $L_X \mu = (div_\mu X)\mu$.

$$(E_{top}) \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) &= grad\ p \\ div_\mu\, u(t) &= 0 \end{cases}$$

Theorem: (Eichhorn, Schmid) Assume (M^n, g) open with (I) and (B_k), $k - 2 \geq r > \frac{n}{2} + 1$.
Then $u(t)$ satisfies the topological Euler equations (E_{top}) iff $\{f_t\}_t$ is a geodesic in $Diff_\mu^{\infty,r}(M)$.

2. KdV equation and the group of Fourier integral operators

Korteweg deVries (KdV) equation $\quad u_t = 6uu_x - u_{xxx}$

Poisson bracket $\quad \{F, G\}(u) = \int \frac{\delta F}{\delta u} \partial_x \frac{\delta G}{\delta u} dx$

Hamiltonian $\quad H(u) = \int (u^3 + \frac{1}{2} u_x^2) dx$

Hamilton's equations $\quad u_t = \{u, H\} \iff u$ satisfies KdV

Gardner, Kruskal 1971

Theorem: (*M.Adams, J.Eichhorn, T.Ratiu, R.Schmid*)
A: The KdV equation is a Hamiltonian system with respect to the Lie-Poisson bracket on the coadjoint orbit of the Lie group of invertible Fourier integral operators $\mathbf{G} = FIO_*$ through the Schrödinger operator.

B: The Kostant-Symes theorem applied to a splitting of the Lie algebra of FIO_* , the space of pseudodifferential operators $\mathbf{g} = \Psi DO$ gives the complete integrability of KdV, i.e. the Gelfand-Dikii family of commuting integrals, including H.

Consider $M = S^1$ the unit circle. Then each pseudodifferential operator $P \in \Psi DO_m(S^1)$has total symbol of the form $p(x,\xi) = \sum_{-\infty<j\leq m} p_j(x)\xi^j$.

The Lie algebra $\mathbf{g} = \Psi DO$ decomposes into the two subalgebras $\mathbf{h} = \Psi DO_- = \cup_{m<0}\Psi DO_m$ and $\mathbf{k} = \Psi DO_+ = \cup_{m\geq 0}\Psi DO_m$,i.e. $\Psi DO = \Psi DO_- \oplus \Psi DO_+$

The inner product $< P, Q >= Trace(P \cdot Q)$ where $Trace(P) = \int p_{-1}(x)dx$ identifies $\Psi DO^* \simeq \Psi DO$ and $(\Psi DO_-)^* \simeq \Psi DO_+$

So for $\mathbf{g} = \mathbf{h} \oplus \mathbf{k}$ we get $\mathbf{g}^* = \mathbf{k}^\perp \oplus \mathbf{h}^\perp$, i.e

$$\mathbf{g}^* = \Psi DO^* \simeq \Psi DO_+^\perp \oplus \Psi DO_-^\perp \simeq \Psi DO_-^* \oplus \Psi DO_-^\perp \simeq$$
$$\simeq \Psi DO_+ \oplus \Psi DO_-^\perp$$

The Lie-Poisson bracket on $\mathbf{h}^* = \Psi DO_-^* \simeq \Psi DO_+$ at $A \in \Psi DO_+$ is

$$\{F, H\}(A) =< A, [\frac{\delta F}{\delta A}, \frac{\delta H}{\delta A}] >= \int (A \circ [\frac{\delta F}{\delta A}, \frac{\delta H}{\delta A}])_{-1} dx$$

$...)_{-1}$ means order(-1) part of the symbol.

The Lie-Poisson evolution equations $\dot{F} = \{F, H\}$ for any function F on ΨDO_-^* are equivalent to

$$\dot{A} = X_H(A) = ad^*_{\frac{\delta H}{\delta A}}(A) = [\frac{\delta H}{\delta A}, A]_+$$

on $\Psi DO_-^* \simeq \Psi DO_+$, where ...$]_+$ means taking only the part in ΨDO_+

For the Schrödimger operator $A \in \Psi DO_+$ with total symbol $a(x,\xi) = a(x) + \xi^2$ the Lie-Poisson bracket of two functions $F, G : \Psi DO^* \to \mathbf{R}$ at A becomes

$$\{F, G\} = \int \frac{\delta F}{\delta a} \partial_x \frac{\delta G}{\delta a} dx$$

which is the Gardner bracket.

For the Hamiltonian $H = \int (a^3 + \frac{1}{2} a_x^2) dx$ Hamiltons equations $\dot{A} = [\frac{\delta H}{\delta A}, A]_+$ become

$$a_t = 6 a a_x - a_{xxx}$$

which is the KdV equation.

For the functionals
$H_k(A) = Trace(A^k) = \int (A^k)_{-1} dx,\ k \in \mathbf{N}$ we have

$\frac{\delta H_k}{\delta A} = kA^{k-1}$, hence $[A, \frac{\delta H_k}{\delta A}] = [A, kA^{k-1}] = 0$.

Thus H_k are constant on coadjoint orbits. By Kostant-Symes theorem, restricting the H_k to $\Psi DO_-^* \simeq \Psi DO_+$ gives the Gelfand-Dikii family of commuting integrals for KdV.

Ex.

$H_0 = \int a\ dx$

$H_1 = \int \frac{1}{2} a^2\ dx$

$H_2 = \int (a^3 + \frac{1}{2} a_x^2) dx \equiv H$ above !

$H_3 = \int (\frac{5}{8} a^4 + \frac{5}{4} a a_x^2 + \frac{1}{8} a_{xx}^2) dx$

etc.