INFINITE DIMENSIONAL HAMILTONIAN SYSTEMS WITH SYMMETRIES

$(P, \{.,.\})$ Poisson manifold, $\{.,.\} : \mathcal{F}(P) \times \mathcal{F}(P) \to \mathcal{F}(P)$
$\mathcal{F}(P)$ Lie algebra (bilinear, skew, Jacobi identity)
derivation i.e $\{F \cdot G, H\} = F \cdot \{G, H\} + G \cdot \{F, H\}$

For $H \in \mathcal{F}(P)$ define the Hamiltonian vector field

$$X_H(F) = \{F, H\}, \quad F \in \mathcal{F}(P)$$

Hamilton's equations of motion = flow of X_H

$$\dot{F} = X_H(F) = \{F, H\}$$

H = Hamiltonian (energy) function

Classical Examples

1) $P = \mathbf{R}^{2n}$

$$\{F, H\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} - \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q^i}, \; \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}$$

Harmonic oscillator: $H = \frac{1}{2}(q^2 + p^2), \quad \dot{q} = p, \; \dot{p} = -q$

2) $P = V \times V^*$, Banach , pairing $< \varphi, \pi >$, $\varphi \in V, \pi \in V^*$
$\{F, H\} =< \frac{\delta F}{\delta \pi}, \frac{\delta H}{\delta \varphi} > - < \frac{\delta H}{\delta \pi}, \frac{\delta F}{\delta \varphi} >$, $\dot{\varphi} = \frac{\delta H}{\delta \pi}$, $\dot{\pi} = - \frac{\delta H}{\delta \varphi}$

Application: Wave equations
$V = \mathcal{F}(\mathbf{R}^3), V^* = Den(\mathbf{R}^3)$, $< \varphi, \pi >= \int \varphi(x)\pi(x)dx$
$H(\varphi, \pi) = \int (\frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\varphi|^2 + F(\varphi))\ dx$

$\dot{\varphi} = \pi$, $\dot{\pi} = \nabla^2\varphi - F'(\varphi) \Rightarrow \frac{\partial^2\varphi}{\partial t^2} = \nabla^2\varphi - F'(\varphi)$

e.g. $F = 0 \Rightarrow$ linear wave equation

$F = \frac{1}{2}m\varphi^2 \Rightarrow \nabla^2\varphi - \frac{\partial^2\varphi}{\partial t^2} = m\varphi$ Klein Gordon equation

3) $P = T^*Q$ cotangent bundle,
locally $\mathbf{R}^{2n}$ finite dim. and locally $V \times V^*$ infinite dim.

4) (P, ω) symplectic manifold, ω closed, nondegenerate 2-form
$\omega = dp_i \wedge dq^i = d\theta$, $\theta = p_i \wedge dq^i$ in $\mathbf{R}^{2n}$ and locally in T^*Q.
$\omega((\varphi_1, \pi_1), (\varphi_2, \pi_2)) =< \varphi_1, \pi_2 > - < \varphi_2, \pi_1 >$ on $V \times V^*$

$$\{F, H\} = \omega(X_F, X_H)$$

note: A) if P is finite dim. then P is even dim.
B) if $\{.,.\}$ is nondegenerate then $\{.,.\}$ comes form ω

5) **Lie-Poisson bracket** on $\mathbf{g}^*$ dual of a Lie algebra $\mathbf{g}$.

G Lie group, Lie algebra $\mathbf{g} = T_eG \simeq$ left invariant vector fields on G , $[.,.]$ Lie bracket on $\mathbf{g}$
dual pairing $< .,. >$: $\mathbf{g}^* \times \mathbf{g} \to \mathbf{R}$, $F, H \in \mathcal{F}(\mathbf{g}^*), \mu \in \mathbf{g}^*$

$$\{F,H\}(\mu) = - < \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}] >, \quad \frac{\delta F}{\delta \mu} \in \mathbf{g}$$

Lie-Poisson bracket is degenerate, e.g. $G = SO(3)$.

Rigid body: $G = SO(3), \quad \mathbf{g}^* = so(3)^* \simeq \mathbf{R}^3$

$H = \frac{1}{2}(\frac{m_1^2}{I_1^2} + \frac{m_2^2}{I_2^2} + \frac{m_3^2}{I_3^2}), \quad \dot{m}_1 = \frac{I_2 - I_3}{I_2 I_3} m_2 m_3$, **Euler equ.**

These are the canonical examples of Poisson bracket, more from symmetries

Reduction Theorem(Marsden-Weinstein)
For a Hamiltonian action of a Lie group G on a Poisson manifold $(P, \{.,.\})$, there is an equivariant momentum map $J : P \to \mathbf{g}^*$ and every reduced pase space $P_\mu = J^{-1}(\mu)/G_\mu$ carries an induced Poisson structure $\{.,.\}_\mu$. For any G-invariant Hamiltonian H on P the integral curves of X_H project onto integral curves of the induced $\hat{X}_{H_\mu}$ on the reduced space P_μ.

Rigid body: If $P = T^*G$ and G is acting on T^*G by cotangent lift of left translation, then the momentum map $J : T^*G \to \mathbf{g}^*$ is $J(\alpha_g) = T_e^* R_g(\alpha_g)$ and $(T^*G)_\mu = J^{-1}(\mu)/G_\mu \simeq \mathcal{O}_\mu$ coadjoint orbit through $\mu \in \mathbf{g}^*$, in particular $T^*G/G \simeq \mathbf{g}^*$

Infinite dimensional Lie groups

$\mathcal{G}$ group and infinite dimensional manifold with smooth group operations

$$m : \mathcal{G}\times\mathcal{G} \to \mathcal{G} : m(g,h) = g\cdot h \ , \ i : \mathcal{G} \to \mathcal{G} : i(g) = g^{-1} \ C^\infty$$

$\mathcal{G}$ is locally ∞ dim. vector space
Banach $\|.\|$, Hilbert $< .,. >$, Frechet metric, no norm

Lie aglebra $\mathbf{g} = T_e\mathcal{G} \simeq$ left invariant vector fields on $\mathcal{G}$
$\xi \in T_e\mathcal{G} \mapsto X_\xi(g) = T_eL_g(\xi)$, $[\xi, \eta] = [X_\xi, X_\eta](e)$

Classical results in finite dimensions which are **NOT** true in infinite dimensions:

1) Exponential map:
$exp : \mathbf{g} \to G : \xi \in \mathbf{g} \mapsto X_\xi$, flow $\varphi_\xi(t)$, $exp(\xi) = \varphi_\xi(1)$ local diffemomorphism $\Rightarrow$ canonical coordinates

2) $f_1, f_2 : G_1 \to G_2$ smooth homomorphisms $T_e f_1 = T_e f_2$ $\Rightarrow f_1 = f_2$ locally

3) $H \subset G$ closed subgroup $\Rightarrow$ H Lie subgroup

4) $\mathbf{g}$ any finite dim. Lie algebra. There exists a connected Lie group G such that $\mathbf{g} \simeq T_e G$

Finite dim. examples: matrix groups
$GL(n),\ SL(n),\ O(n),\ SO(n),\ U(n),\ SU(n),\ Sp(n)$.

Infinite dimensional examples:

1) M finite dimensional manifold
$\mathcal{G} = C^\infty(M),\ m(f,g) = f + g,\ i(f) = -f,\ e = 0$
$\mathbf{g} = T_e C^\infty(M) \simeq C^\infty(M),\ [\xi, \eta] = 0$, abelian, $exp = id$
C^∞-Frechet Lie group
complete: $C^k(M)$ -norm, $k < \infty \Rightarrow$ Banach Lie group
H^s-Sobolev norm, $s > \frac{1}{2} dim M \Rightarrow$ Hilbert Lie group

Application: Maxwell's equations
E, B electric and magnetic fields

$$\dot{E} = curl\ B,\ \dot{B} = -curl\ E,\ div\ B = 0,\ div\ E = \rho$$

$V =$ vector fields (potentials) on $\mathbf{R}^3$
$P = T^*V = V \times V^* \ni (A, E)$
L^2 pairing $< A, E >= \int A(x)E(x)dx$,

$\{F, H\}(A, E) = \int (\frac{\delta F}{\delta A}\ \frac{\delta H}{\delta E} - \frac{\delta H}{\delta A}\ \frac{\delta F}{\delta E})dx$

energy $H(A, E) = \frac{1}{2} \int (|B|^2 + |E|^2)dx, \quad B = -\ curl\ A$

Hamiltons equations $\dot{A} = \frac{\delta H}{\delta E} = E \Rightarrow \dot{B} = -curl\ E$
$\dot{E} = -\frac{\delta H}{\delta A} = -curl\ curl A = curl B,\ div B = div\ curl\ A = 0.$

$div E = \rho$ from symmetry

Gauge invariance: $\mathcal{G} = C^\infty(\mathbf{R}^3)$ acts on V by
$\varphi \cdot A = A + \nabla\varphi$
lifted action to $V \times V^*$, $\varphi \cdot (A, E) = (A + \nabla\varphi, E)$
H is $\mathcal{G}$ invariant
momentum map $J : V \times V^* \to \mathbf{g}^* \simeq$charge densities

$$J(A, E) = div\ E$$

reduced pase space for $\rho \in \mathbf{g}^*$
$(V \times V^*)_\rho = J^{-1}(\rho)/G = \{(E, B) | div\ E = \rho,\ div\ B = 0\}$

reduced Hamiltonian:

$$H_\rho(E, B) = \frac{1}{2} \int (|E|^2 + |B|^2) dx$$

reduced bracket:

$$\{F, H\}_\rho(E, B) = \int (\frac{\delta F}{\delta E} \cdot curl\ \frac{\delta H}{\delta B} - \frac{\delta H}{\delta E} \cdot curl\ \frac{\delta F}{\delta B}) dx$$

$$\dot{F} = \{F, H_\rho\}_\rho \Leftrightarrow \begin{cases} \dot{E} = curl\ B & , \quad \dot{B} = -curl\ E \\ div\ B = 0 & , \quad div\ E = \rho \end{cases}$$

Maxwell's equations

2) M finite dimensional manifold
$\mathcal{G} = C^\infty(M, \mathbf{R} - \{0\}),\ m(f,g) = f \cdot g,\ i(f) = f^{-1},\ e = 1$
$C^k(M, \mathbf{R} - \{0\})$ open in $C^\infty(M, \mathbf{R})$
M compact $\Rightarrow$ Banach Lie group
M compact $\Rightarrow H^s(M, \mathbf{R} - \{0\})$ closed under multiplication if $s > \frac{1}{2} dim M$, Hilbert Lie group

3) generalize $\mathbf{R} - \{0\}$ to G finite dimensional Lie group
$\mathcal{G} = C^k(M, G),\ m(f,g)(x) = f(x) \cdot g(x)$ pointwise
$\mathbf{g} = C^k(M, g),\ [\xi, \eta](x) = [\xi(x), \eta(x)]$ pointwise

$EXP : \mathbf{g} = C^k(M, g) \to \mathcal{G} = C^k(M, G),\ EXP(\xi) = exp \circ \xi$
local diffeomorphism

Applications: Gauge theories, quantum field theory

Special case: $M = S^1$ circle

$\mathcal{G} = C^k(S^1, G) = L^k(G)$ loop group
$\mathbf{g} = C^k(S^1, g) = l^k(g)$ loop algebra

Applications: affine Lie algebras, Kac-Moody Lie algebras (central extensions)
- completely integrable systems
- soliton equations (Toda, KdV, KP)
- quantum field theory

Diffeomorphism groups

M compact manifold
(noncompact in progress, J. Eichhorn)
$\mathcal{G} = Diff^{\infty}(M),\ m(f,g) = f \circ g,\ i(f) = f^{-1},\ e = id_M$
Frechet manifold; meaning of C^{∞} ?
generalization to notion of C^{∞}_{Γ} differentiability or

$$\begin{aligned} Diff^{\infty}(M) &= \lim_{\leftarrow} Diff^k(M), \quad ILB \\ &= \lim_{\leftarrow} Diff^s(M), \quad ILH \end{aligned}$$

H^s -top: $Diff^s(M),\ C^{\infty}$ - Hilbert manifold if $s > \frac{1}{2} dim M$

$m : Diff^{s+k}(M) \times Diff^s(M) \to Diff^s(M) \qquad C^k$
$k = 0$: m is only continuous
$i : Diff^{s+k}(M) \to Diff^s(M) \quad C^k, \quad k = 0$ continuous

$\Rightarrow$ notion of nested Lie groups (Adams, Ratiu, Schmid; Omori)

Lie algebra: $\mathbf{g} = T_e Diff(M) \simeq Vec(M)$ vector fields on M. Lie algebra only for C^{∞} vector fields.

$EXP : Vec(M) \to Diff(M) : X \mapsto \varphi_1$ flow at $t = 1$.
NOT local diffeomorphism, not locally onto !

Applications of $Diff(M)$:

1) **General relativity**: (M, g) Lorentz 4 -manifold
Einstein's field equations $Ric(g) = 0$ are
invariant under $Diff(M)$ (coordinate transf.)
Hamiltonian system on $P = \{metrics\}/Diff(M)$
(Fischer, Marsden)

Subgroups of $Diff(M)$:
2) μ = volume on M : $\mathcal{G} = Diff_\mu(M) = \{f \mid f^*\mu = \mu\}$
volume preserving diffeomorphisms
$Diff_\mu(M)$ closed subgroup of $Diff(M)$
Lie algebra: $\mathrm{g} = Vec_\mu(M) = \{X \mid div_\mu X = 0\}$
divergence free vector fields
$Vec_\mu(M)$ Lie subalgebra of $Vec(M)$

Note: cannot apply $Vec_\mu(M)$ Lie algebra $\Rightarrow$ ex. Lie group
$Diff_\mu(M) \subset Diff(M)$ closed subgroup $\Rightarrow$ Lie subgroup

Applications: Fluid dynamics
Euler's equations for incompressible fluid

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p \ , \qquad div \ u = 0$$

are equivalent to geodesics on $Diff_\mu(M)$
(Arnold, Ebin, Fischer, Marsden)

3) $\omega =$ symplectic 2-from on M
$\mathcal{G} = Diff_\omega(M) = \{f \mid f^*\omega = \omega\}$
canonical transformations, symplectomorphisms
Lie algebra: $\mathbf{g} = Vec_\omega(M) = \{X \mid L_X\omega = 0\}$
locally Hamiltonian vector fields

Applications: Plasma physics
Maxwell-Vlasov equations for density $f(x, v, t)$

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + (E + v \times B)\frac{\partial f}{\partial v} = 0$$

$$\frac{\partial B}{\partial t} = -curl\ E$$

$\frac{\partial E}{\partial t} = curl\ B - J_f$, J_f= current density

$div\ E = \rho_f$, ρ_f = charge density

$$div\ B = 0$$

This coupled non-linear system of evolution equations is Hamiltonian $\dot{F} = \{\{F, H\}\}$ on the reduced phace space

$\mathcal{MV} = (T^*Diff_\omega(\mathbf{R}^6) \times T^*V)/C^\infty(\mathbf{R}^6)$ with respect to the reduced Poisson bracket

$$\{\{F,G\}\}(f,E,B) = \int f\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\} dxdv$$

$$+ \int (\frac{\delta F}{\delta E} curl \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} curl \frac{\delta F}{\delta B}) dxdv$$

$$+ \int (\frac{\delta F}{\delta E} \cdot \frac{\partial f}{\partial v} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta E} \cdot \frac{\partial f}{\partial v} \frac{\delta F}{\delta f}) dxdv$$

$$+ \int f \cdot B (\frac{\partial}{\partial v} \frac{\delta F}{\delta f} \times \frac{\partial}{\partial v} \frac{\delta G}{\delta f}) dxdv$$

with Hamiltonian

$$H(f,E,B) = \frac{1}{2} \int v^2 f(x,v,t) dv + \frac{1}{2} \int (|E|^2 + |B|^2) dx$$

(Marsden, Ratiu, Schmid, Weinstein)

More complicated plasma models:

- two fluid model: phase space = coadjoint orbits of

$$\mathcal{G} = Diff^{\infty}(\mathbf{R}^3) \odot (C^{\infty}(\mathbf{R}^3) \times C^{\infty}(\mathbf{R}^3))$$

-MHD: $\mathcal{G} = Diff^{\infty}(\mathbf{R}^3) \odot (C^{\infty}(\mathbf{R}^3) \times \Omega^2(\mathbf{R}^3))$

Fourier integral operators , KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad KdV$$

Gardner:

$$\{F,G\} = \int_0^{2\pi} \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta G}{\delta u}dx, \quad H(u) = \int_0^{2\pi}(u^3 + \frac{1}{2}u_x^3)dx$$

$$\dot{u} = \{u, H\} \ \Leftrightarrow \quad u \ satisfies \ KdV$$

The symmetry goup for KdV is $\mathcal{G} = FIO$ invertible Fourier integral operators on the circle S^1.

$FIO: \quad A : C^\infty(M) \to C^\infty(M)$ locally given by

$$A(u)(x) = (2\pi)^{-n} \int\int e^{i\varphi(x,y,\xi)} a(x,\xi)u(y)dyd\xi$$

pseudodifferential operator ΨDO

$$P(u)(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} p(x,\xi)u(y)dyd\xi$$

Theorem: (Adams, Ratiu, Schmid)
a) FIO is a smooth ∞ dim. ILH - Lie group
b) FIO is a smooth ∞ dim. principal fiber bundle over $Diff_\omega(T^*M - \{0\})$ with structure group ΨDO
Theorem: (Adams, Ratiu, Schmid) *The Gardner bracket is the Lie Poisson bracket on the coadjoint orbit of FIO through the Schroedinger operator. Complete integrability.*

Wess-Zumino Consistency Condition for BRST Symmetries

Quantum field theory problem in local Lie algebra cohomology.
Lie group $\mathcal{G}$ = group of gauge transformations on a principal bundle (P, M, G), $\mathcal{G} = C^\infty(P, G)$. Lie algebra $\mathbf{g} = C^\infty(P, g)$ infinitesimal gauge transformations

bicomplex $\mathcal{C}^*_{loc} = \{\mathcal{C}^{q,p}, \Delta\}_{q,p\in\mathbf{N}}$, $\mathcal{C}^{q,p} = \mathcal{C}^q(\mathbf{g}, \Omega^p_{loc}(P, g))$

total differential $\Delta = \delta_{loc} + (-1)^p d$ = BRST operator

$\delta_{loc} : \mathcal{C}^{q,p}(\mathbf{g}, \Omega_{loc}) \longrightarrow \mathcal{C}^{q+1,p}(\mathbf{g}, \Omega_{loc})$
Chevalley-Eilenberg coboundary operator

$d : \mathcal{C}^{q,p}(\mathbf{g}, \Omega_{loc}) \longrightarrow \mathcal{C}^{q,p+1}(\mathbf{g}, \Omega_{loc})$
induced exterior derivative

$$\Delta^2 = \delta_{loc} d + d\delta_{loc} = \delta^2_{loc} = d^2 = 0$$

Wess-Zumino consistency condition for $\omega \in \mathcal{C}^*_{loc}$ means there exists $\alpha \in \mathcal{C}^*_{loc}$ such that

$$\delta_{loc}\omega + d\alpha = 0$$

Gauge anomalies (chiral) are elements in induced BRST cohomology $\mathcal{H}^*_{BRST}(\mathbf{g}, \Omega_{loc})$.

$\mathbf{g}$-symplectic structures

Definition: *A $\mathbf{g}$-symplectic structures on P is a $\mathbf{g}$ -form $\Omega \in \Omega^2(P, \mathbf{g})$ which is closed and nondegenerate. A vector field X on P is called* $\mathbf{g}$-Hamiltonian *if exists* $\mathbf{g}$*-function $f : P \to \mathbf{g}$ such that $df = i_X \Omega$*

A $\mathbf{g}$ - vector field X is locally $\mathbf{g}$ - Hamiltonian iff its flow φ_t is $\mathbf{g}$ - symplectic, i.e $\varphi_t^* \Omega = \Omega$.
Poincare: For any $\alpha \in \Omega^p(\mathbf{R}^n, \mathbf{g})$, $d\alpha = 0$ exists locally $\beta \in \Omega^{p-1}(\mathbf{R}^n, \mathbf{g})$ s.t. $\alpha = d\beta$.

Theorem: *If G is semi simple, then every G - orbit $\mathcal{O}_p$ of the right action of G on P is a $\mathbf{g}$ - symplectic manifold; induced by the Maurer Cartan form on G.*

Canonical Momentum Map on $\mathcal{O}_p$

Proposition: *For every $\xi \in \mathbf{g}$ the fundamental vector field ξ_P on $\mathcal{O}_p$ is locally $\mathbf{g}$ -Hamiltonian.*

Corollary: *The $\mathbf{g}$ -momentum map of the right action of G on $\mathcal{O}_p$ is given by $J : \mathcal{O}_p \to \mathcal{L}(\mathbf{g}, \mathbf{g})$*

$$J(q) = ad_\eta \circ TR_q$$

where $\eta = R_p^* X_t(g)$, $q = p \cdot g$

Solution of Consistency Condition

Infinite dimensional analogue: $\mathcal{G} = C^{\infty}(P, G)$

principal $\mathcal{G}$ bundle $(\mathcal{P}, \pi, \mathcal{M})$, where $\mathcal{P} = \Omega^*(P, \mathbf{g})$, adjoint $\mathcal{G}$ action and $\mathcal{M} = \mathcal{P}/\mathcal{G}$.
For $A \in \Omega^*(P, \mathbf{g})$ the canonical 1-form Θ_A on the orbit $\mathcal{O}_A$ induced from the Maurer - Cartan form on $\mathcal{G}$ becomes a map

$$\Theta_A : \mathcal{O}_A \to \Omega^1(P, \mathbf{g}) \simeq \mathcal{C}^{0,1}_{loc}$$

and the momentum map

$$J : \mathcal{O}_A \to \mathcal{L}(\mathbf{g}, \mathbf{g}) = \mathcal{C}^{1,0}_{loc}$$

Theorem : *The momentum map J satisfies the consistency condition for the canonical 1-form (Maurer-Cartan) Θ of $\mathcal{G}$*

$$\delta_{loc}\Theta_A + dJ = 0.$$

They represent the ghost and the chiral anomaly in $\mathcal{H}^1_{BRST}(\mathbf{g}, \Omega_{loc})$.