

THE LIE GROUP OF FOURIER INTEGRAL OPERATORS * AND APPLICATIONS TO HYDRODYNAMICS

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The Lie Group of Fourier Integral Operators on **open Manifolds*,
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COMPACT case:

Theorem: (M.Adams, T.Ratiu, R.Schmid, 1985)

The group $FIO_*(M)$ of invertible Fourier integral operators on a compact manifold M is a graded ILH-Lie group with graded ILH-Lie algebra $\Psi DO(M)$ of pseudodifferential operators on M .

IDEA:

$FIO_*(M)$ is an ∞ -dim principal fiber bundle over the base manifold $Diff_\theta(\dot{T}^*M)$ of contact transformations of \dot{T}^*M with gauge group $\Psi DO_*(M)$ of invertible pseudodifferential operators.

$$\begin{array}{ccc} \Psi DO_*(M) & \longrightarrow & FIO_*(M) \\ & & \downarrow \\ & & Diff_\theta(\dot{T}^*M) \end{array}$$

- FIO , Fourier integral operators, they generalize
- ΨDO , pseudodifferential operators, they generalize
- DO , differential operators

$$FIO \supset \Psi DO \supset DO$$

$P \in \Psi DO_m$ classical ΨDO order m , locally : $u \in C_c^\infty(M)$

$$Pu(x) = (2\pi)^{-n} \int \int e^{i(x-y) \cdot \xi} p(x, \xi) u(y) dy d\xi$$

- $p(x, \xi)$ classical symbol of order m

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi), \quad p_{m-j}(x, t\xi) = t^{m-j} p_{m-j}(x, \xi)$$

- special case:

$$p(x, \xi) = \sum_{|\alpha| \leq m} p_\alpha(x) \xi^\alpha \quad \text{polynomial in } \xi$$

$\Rightarrow P \in DO$ differential operator order m

P oscillatory integral, highly singular, nice operators:

(i) invariant under diffeomorphisms \Rightarrow def. on M

(ii) extend $P : H_c^s(M) \rightarrow H_c^{s-m}(M)$ bounded

(iii) closed under composition, order additive

(iv) $\Psi DO = \bigcup_m \Psi DO_m$ is an ∞ -dim graded Lie algebra

$P \in \Psi DO_m, Q \in \Psi DO_n \Rightarrow [P, Q] = PQ - QP \in \Psi DO_{m+n-1}$

Note: ΨDO_1 is an ∞ -dim Lie algebra

Question: Is there an ∞ -dim Lie group which has ΨDO or ΨDO_1 as Lie algebra?

Answer: YES $\mathcal{G} = FIO_*$ and $\mathcal{G}_o = (FIO_o)_*$ the groups of invertible Fourier integral operators (order zero)

$A \in FIO_m$ Fourier integral operator of order m on M compact, locally:

$$Au(x) = (2\pi)^{-n} \int \int e^{i\varphi(x,y,\xi)} a(x,\xi) u(y) dy d\xi$$

- $a(x,\xi)$ classical symbol of order m
- $\varphi(x,y,\xi)$ nondegenerate phase function

Note: if $\varphi(x,y,\xi) = (x-y) \cdot \xi \Rightarrow A \in \Psi DO_m$

φ is locally generating a homogeneous canonical transformation $\eta : \dot{T}^*M \rightarrow \dot{T}^*M : \eta^*\omega = \omega, \eta(t\alpha) = t\eta(\alpha)$
 $\Leftrightarrow \eta^*\theta = \theta$ canonical 1-form

- (i) $A : C_c^\infty(M) \rightarrow C^\infty(M)$ extends $A : \mathcal{E}'(M) \rightarrow \mathcal{D}'(\mathcal{M})$
(ii) closed under composition, order additive, and if $A = A_1 \circ A_2$ then $\eta = \eta_1 \circ \eta_2$, and $A^{-1} \Leftrightarrow \eta^{-1}$

Let $FIO = \bigcup_m FIO_m$, FIO_* group of invertible FIO_s
 $Diff_\theta^\infty = \{\eta \in Diff^\infty(\dot{T}^*M) \mid \eta^*\theta = \theta\}$

$p : FIO_* \rightarrow Diff_\theta^\infty$, $p(A) = \eta$, p is surjective and $ker(p) = \Psi DO_*$, since $\varphi(x,y,\xi) = (x-y) \cdot \xi \Rightarrow id$

i.e. we have an exact sequence of groups

$$I \rightarrow \Psi DO_* \xrightarrow{j} FIO_* \xrightarrow{p} Diff_\theta^\infty \rightarrow id$$

We want : LIE GROUPS
zero order groups

$$I \rightarrow (\Psi DO_0)_* \xrightarrow{j} (FIO_0)_* \xrightarrow{p} Diff_\theta^\infty \rightarrow id$$

Lie algebras

$$0 \longrightarrow \Psi DO_0 \hookrightarrow \Psi DO_1 \longrightarrow Vec_\theta^\infty \longrightarrow 0$$

Note: $Diff_\theta^\infty \neq exp(Vec_\theta^\infty)$, $FIO \neq exp(\Psi DO)$

Idea: construct ∞ -dim principal fiber bundle such that

- base space = $Diff_\theta^\infty(T^*M)$
- total space = $(FIO_0)_*$
- fiber = $p^{-1}(\eta) \simeq (\Psi DO_0)_* =$ gauge group

step 1: $Diff_\theta^\infty = \varprojlim Diff_\theta^s$ is an ILH Lie group

step 2: $(\Psi DO_0)_* = \varprojlim (\Psi DO_0^s)_*$ is an ILH Lie group

step 3: piece 1 & 2 together via local section σ

$$\sigma : U \subset Diff_\theta^\infty \rightarrow (FIO_0)_*$$

then $(FIO_0)_*$ locally: $p^{-1}(U) \simeq \sigma(U) \times (\Psi DO_0)_*$

\Rightarrow chart at identity $I \in (FIO_0)_*$

step 4: move this chart around by group structure of
 $Diff_\theta^\infty$

step 5: chart transitions smooth $\Rightarrow (FIO_0)_*$ smooth manifold

step 6: multiplication "smooth" $\Rightarrow (FIO_0)_*$ Lie group

step 7: identify $(1 - \Delta)^{m/2} : (FIO_0)_* \xrightarrow{\sim} (FIO_m)_*$

Theorem: (M. Adams, T. Ratiu, R. Schmid, 1985) :
The group $FIO_*(M)$ of invertible Fourier integral operators on a compact manifold M is a graded ILH-Lie group with graded ILH-Lie algebra $\Psi DO(M)$ of pseudodifferential operators on M .

$FIO_*(M)$ is an ∞ -dim principal fiber bundle over the base manifold $Diff_\theta^s(\dot{T}^*M)$ of contact transformations of \dot{T}^*M with gauge group $\Psi DO_*(M)$ of invertible pseudodifferential operators.

NON-COMPACT case

J. Eichhorn: "There is exactly one thing that works in the non-compact case: NOTHING"

Diffeomorphisms of NON-COMPACT manifolds:

Example of what's going wrong:

Let M, N be compact manifolds, then $f : M \rightarrow N$ is of Sobolev class $H^s \iff$ the local representatives

$f_j^i : U_i \subset \mathbf{R}^m \rightarrow V_j \subset \mathbf{R}^n$ are of class H^s ,

where $M = \cup(U_i, \phi_i)$, $N = \cup(V_j, \psi_j)$, $f_j^i := \psi_j \circ f \circ \phi_i^{-1}$.

These covers are **finite** if M, N are compact.

This definition is invariant $\iff s > \frac{n}{2} + 1$.

This definition is meaningless if M, N are open !

Idea: Bounded Geometry

- Control over the metric and its derivatives
- Control over the mappings and their derivatives by the metric i.e. maps adapted to the bounded geometry

Definition: (M^n, g) has *bounded geometry of order k* , $0 \leq k \leq \infty$, if it has a positive injectivity radius and the curvature and all its derivatives up to order k are uniformly bounded; i.e the following two conditions are satisfied:

$$(I) : r_{inj}(M) = \inf_{x \in M} r_{inj}(x) > 0$$

$$(B_k) : |\nabla^i R| \leq C_i, \quad 0 \leq i \leq k.$$

(I) \Leftrightarrow there exists a ball around 0 in \mathbf{R}^n which is domain of normal (geodesic) coordinates **for all** $x \in M$.

(B_k) \Leftrightarrow there exists a constant d_k (independent of $x \in M$) such that $\|g_{ij}\|_{C^k} \leq d_k$ in any normal coordinate system
 $\Leftrightarrow \|\Gamma_{ij}^m\|_{C^{k-1}} \leq d_k$ in any normal coordinate system

Examples of manifolds with bounded geometry:
 compact manifolds, Lie groups, homogeneous spaces, covering spaces of Riemannian manifolds, leaves of foliations of compact manifolds.

Fact: There is **no** topological obstruction for a complete metric with bounded geometry of any order.

Bounded maps $C^{\infty,r}(M, N)$:

Let $(M, g), (N, h)$ open, complete Riemannian manifolds satisfying (I) , (B_k) and $f \in C^\infty(M, N)$. Assume $r \leq k$. Let $C^{\infty,r}(M, N)$ be the set of all $f \in C^\infty(M, N)$ satisfying $|df|_r := \sum_{i=0}^{r-1} \sup_{x \in M} |\nabla^i df|_x < \infty$.

Equivalently: $f \in C^{\infty,r}(M, N) \Leftrightarrow \frac{\partial^\alpha}{\partial x^\alpha} f^\nu$ is uniformly bounded in any normal coordinate system;

$|\alpha| \leq r, 1 \leq r \leq k$.

Topology: $C^{p,r}(M, N)$ the completion of $C^{\infty,r}(M, N)$ in H^p Sobolev topology is a C^{k+1-r} -Banach manifold, $p = 2$ Hilbert manifold.

The bounded diffeomorphism group $Diff^{p,r}(M)$.

additional assumption to obtain a group:

Set $Diff^{p,r}(M) := \{f \in C^{p,r}(M, M) \mid f \text{ bijective, preserves orientation and } |\lambda|_{\min}(df) > 0\}$,

then $Diff^{p,r}(M)$ is open in $C^{p,r}(M, M)$, hence a C^{k+1-r} -Banach manifold

Theorem:(J. Eichhorn, R. Schmid)

Let (M^n, g) be an open, oriented, complete Riemannian manifold satisfying (I), (B_∞) and let $r > \frac{n}{p} + 1$. Then $Diff^{p,\infty}(M) = \lim_{\leftarrow} Diff^{p,r}(M)$ is an ILB - Lie group; and for $p = 2$ it is an ILH - Lie group.

Volume preserving and symplectic diffeomorphisms

Let ω be a C^∞ -bounded strongly nondegenerate closed q -form and let $Diff_\omega^r = \{f \in Diff^r \mid f^*\omega = \omega\}$.

Then for $q = n$ or $q = 2$ $Diff_\omega^{p,\infty}(M) = \lim_{\leftarrow} Diff_\omega^{p,r}(M)$ is an ILH-Lie group with Lie algebra consisting of all divergence free ($q = n$), or locally Hamiltonian ($q = 2$) H^s vector fields.

Pseudodifferential operators and Fourier integral operators on open manifolds

If (M^n, g) is open the previous definition of ΨDO_s and FIO_s does not make sense. We need to adapt the class of symbols and phase functions to the bounded geometry of M . Then the corresponding spaces ΨDO and FIO have similar properties as in the compact case and we can use the same ideas as before to construct Lie group structures (technical more complicated!).

$$FIO \quad Au(x) = (2\pi)^{-n} \int \int e^{i\varphi(x,y,\xi)} a(x,\xi) u(y) dy d\xi$$

- **symbols:** the family of local symbols together with their derivatives should be uniformly bounded
- **phase functions:** the phase functions should locally generate canonical transformations in the space $Diff_{\theta}^{p,r}(T^*M)$

FINAL MAIN RESULT

$\Psi DO = \varprojlim \Psi DO^s$ is an ILH Lie group
 $FIO = \varprojlim FIO^t$ is an ILH Lie group

multiplication $\mu : FIO^{t+r} \times FIO^t \rightarrow FIO^t ; \mu(A, B) = AB$
is C^k differentiable, $k = \min(r, t)$

inversion $\nu : FIO^{t+r} \rightarrow FIO^t ; \nu(A) = A^{-1}$
is C^k differentiable, $k = \min(r, t)$

right mult. $R_A : FIO^t \rightarrow FIO^t ; R_A(B) = BA$
is C^t differentiable, for any $A \in FIO^t$

left mult. $L_A : FIO^t \rightarrow FIO^t ; L_A(B) = AB$
is C^0 (continuous), for any $A \in FIO^t$

APPLICATIONS to HYDRODYNAMICS

1. Euler equations and $Diff_{\mu}^{\infty,r}(M)$

Topological Euler equations

$$E_{top} \begin{cases} \frac{\partial u}{\partial t} + \nabla_{u(t)} u(t) = \text{grad } p \\ \text{div}_{\mu} u(t) = 0 \end{cases}$$

$\mu =$ fixed volume form on (M^n, g)

$u = u(x, t)$ is a time dependent C^1 vector field on (M^n, g)

$\nabla = \nabla^g$ the Riemannian covariant derivative

but $\text{div} = \text{div}_{\mu}$, defined by $L_X \mu = (\text{div}_{\mu} X) \mu$.

Then $u(t)$ defines a 1-parameter family of diffeomorphisms f_t defined by

$$\left. \frac{df_s}{ds} \right|_{s=t} = u(t) \circ f_t .$$

Theorem:(J.Eichhorn, R. Schmid)

Assume (M^n, g) with (I) and (B_k) . Then $u(t)$ satisfies the topological Euler equations E_{top} iff $\{f_t\}_t$ is a geodesic in $Diff_{\mu}^{\infty,r}(M)$.

Classical Euler equations for an incompressible, homogeneous fluid without viscosity

$$\nabla = \nabla^g, \text{div} = \text{div}_{dvol_x(g)} .$$

2. KdV equation and the group of Fourier integral operators

Korteweg deVries (KdV) equation $u_t = 6uu_x - u_{xxx}$

Poisson bracket $\{F, G\}(u) = \int \frac{\delta F}{\delta u} \partial_x \frac{\delta G}{\delta u} dx$

Hamiltonian $H(u) = \int (u^3 + \frac{1}{2}u_x^2) dx$

Hamilton's equations $u_t = \{u, H\} \iff u$ satisfies KdV
Gardner, Kruskal 1971

Theorem: (*M.Adams, J.Eichhorn, T.Ratiu, R.Schmid*)

A: The KdV equation is a Hamiltonian system with respect to the Lie-Poisson bracket on the coadjoint orbit of the Lie group $G = FIO_*$ of invertible Fourier integral operators

through the Schrödinger operator.

B: The Kostant-Symes theorem applied to the splitting of the Lie algebra \mathfrak{g} of FIO_* , the space of pseudodifferential operators $\mathfrak{g} = \Psi DO = \Psi DO_- \oplus \Psi DO_+$, gives the complete integrability of KdV, i.e. the Gelfand-Dikii family of commuting integrals, including H .