DIFFEOMORPHISM GROUPS OF *OPEN* MANIFOLDS

Rudolf Schmid
Department of Mathematics
Emory University
Atlanta, Georgia 30322, USA
e-mail: rudolf@mathcs.emory.edu

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1. Review COMPACT case:

Let M be a smooth **compact** manifold and $Diff(M) = \{f : M \to M | diffeomorphism\}$ the group of diffeomorphisms . For different classes of diffeomorphisms we have the following classical results:

- diffeomorphisms of class $C^k, k < \infty$
  $Diff^k(M)$ is a Banach manifold
  (Palais 68, Omori 74)

- diffeomorphisms of class C^∞
  $Diff^\infty(M)$ is a Fréchet manifold
  (Omori, Gutknecht, Schmid, 78)

- diffeomorphisms of Sobolev class $H^s, s > \frac{1}{2} dim M$
  $Diff^s(M)$ is a Hilbert manifold
  (Ebin-Fischer-Marsden 70, Ratiu-Schmid 79)

Lie group structure of $Diff(M)$

The group multiplication
$m : Diff^{s+t}(M) \times Diff^{s}(M) \to Diff^{s}(M)$
$m(f, g) = f \circ g$ is of class C^t
e.g if $t = 0$: m is only continuous.
The inversion map
$i : Diff^{s+t}(M) \to Diff^{s}(M)$
$i(f) = f^{-1}$ is of class C^t
e.g. if $t = 0$ i is only continuous.
In the notion of nested Lie groups (Adams, Ratiu, Schmid; Omori) we have the Fréchet Lie group $Diff^{\infty}(M)$ as inverse limits

$$\begin{aligned} Diff^{\infty}(M) &= \lim_{\overleftarrow{k}} Diff^{k}(M), \quad ILB - Lie\ group \\ Diff^{\infty}(M) &= \lim_{\overleftarrow{s}} Diff^{s}(M), \quad ILH - Lie\ group \end{aligned}$$

The **Lie algebra** of $Diff(M)$ is identified as tangent space at the identity $e = id$, $T_e Diff(M) \simeq Vec(M)$ the space of vector fields on M. This is only a Lie algebra (i.e. closed under the bracket) for C^{∞} vector fields.

The **exponential map** $EXP : Vec(M) \to Diff(M) :$ $X \mapsto \Phi_{t=1}$ the flow of X at time $t = 1$ is **NOT** a local diffeomorphism, it is not locally onto !

Important Subgroups of $Diff^{\infty}(M)$:

1) Volume preserving diffeomorphisms

Let μ be a volume form on M and $Diff_{\mu}^{\infty}(M) = \{f \mid f^{*}\mu = \mu\}$ the group of volume preserving diffeomorphisms. Then $Diff_{\mu}^{\infty}(M)$ is an ILH - (ILB) - Lie group (a closed subgroup of $Diff^{\infty}(M)$) with Lie algebra $Vec_{\mu}^{\infty}(M) = \{\xi \in Vec^{\infty}(M) \mid div_{\mu}\xi = 0\}$ the space of divergence free vector fields on M.
($Vec_{\mu}^{\infty}(M)$ is a Lie subalgebra of $Vec^{\infty}(M)$.)

2) Symplectomorphisms

Let ω be a symplectic 2-from on M and $Diff_{\omega}^{\infty}(M) = \{f \mid f^{*}\omega = \omega\}$ the group of canonical transformations, (symplectomorphisms). Then $Diff_{\omega}^{\infty}(M)$ is an ILH - (ILB) - Lie group (a closed subgroup of $Diff^{\infty}(M)$) with Lie algebra: $Vec_{\omega}^{\infty}(M) = \{\xi \in Vec^{\infty}(M) \mid L_{\xi}\omega = 0\}$ the space of locally Hamiltonian vector fields on M.
($Vec_{\omega}^{\infty}(M)$ is a Lie subalgebra of $Vec^{\infty}(M)$.)

2. NON-COMPACT case:

Boos-Bleecker: Topology & Analysis: Atiyah-Singer Index formula & Gauge theoretic Physics:
"*In what follows, the manifolds M is "closed" i.e. compact, without boundary. We make this convention in part for convenience (in order to make some proofs go easier) but also because otherwise some of the following theorems would be meaningless or false.*"
Eichhorn: "*In the non-compact case there is exactly one thing that works: nothing !* "

Example: Let M^n and $N^{n'}$ be open manifolds, then a map $f : M^n \to N^{n'}$ is of Sobolev class H^s if and only if the local representatives $f^i_j : U_i \subset \mathbf{R}^n \to V_j \subset \mathbf{R}^{n'}$ are of class H^s, where $M \subset \bigcup(U_i, \phi_i),\ N \subset \bigcup(V_j, \psi_j),\ f^i_j := \psi_j \circ f \circ \phi_i^{-1}$.
These covers are **finite** if M, N are compact.
This definition is invariant $\Leftrightarrow\ s > \frac{n}{2} + 1$
In the compact case we can define the distance by

$$d^s(f, g) := (\sum_{i,j} \|f^i_j - g^i_j\|^2_s)^{\frac{1}{2}}$$

These definitions are meaningless if M, N are open !

Idea: Bounded Geometry

- Control over the metric and its derivatives

- Control over the mappings and their derivatives by the metric, i.e. maps adapted to the geometry

Definition: A Riemannian manifold (M^n, g) has *bounded geometry of order* k, $0 \leq k \leq \infty$, if M has a positive injectivity radius $r_{inj}(M)$ and the curvature tensor R and all is derivatives up to order k are uniformly bounded; i.e the following two conditions (I) and (B_k) are satisfied:

$$(I): \; r_{inj}(M) = \inf_{x \in M} r_{inj}(x) > 0$$

$$(B_k): \; |\nabla^i R| \leq C_i \; , \; 0 \leq i \leq k.$$

Examples of manifolds with bounded geometry:
- compact manifolds
- Lie groups
- homogeneous spaces
- covering spaces of Riemannian manifolds
- leaves of foliations of compact manifolds

These conditions can be expressed as follows:
(I) The exponential map $exp_x : T_xM \to M$ is a diffeomorphism from an open ball $B_x(0, r) \subset T_xM$ of radius r around 0 in T_xM onto an open neighborhood $U_{x,r} \subset M$ of x in M. Let $r_x := \sup(r)$, i.e. the biggest radius in T_xM such that exp_x is a diffeomorphism. Then the injectivity radius is def. by $r_{inj} := \inf_{x\in M} r_x$, i.e r_{inj} is the smallest distance from x where geodesics intersect.
Hence
$(I) \Leftrightarrow$ there exists a ball around 0 in $\mathbf{R}^n$ which is domain of normal (geodesic) coordinates **for all** $x \in M$.

$(B_k) \Leftrightarrow$ ther exists a constant d_k (independent of $x \in M$) such that $\|g_{ij}\|_{C^k} \leq d_k$ in any normal coordinate system
$\Leftrightarrow |D^\alpha g_{ij}| \leq c_\alpha$, $|\alpha| \leq k$ in any normal coordinate system
$\Leftrightarrow \|\Gamma^m_{ij}\|_{C^{k-1}} \leq d'_k$ in any normal coordinate system
$\Leftrightarrow |D^\beta \Gamma^m_{ij}| \leq c_\beta$, $|\beta| \leq k-1$, in any normal coordinate system.

Fact: Given an open manifold M^n and $k \geq 0$, then there exists a complete Riemannian metric g on M^n satisfying the conditions (I) and (B_k); i.e there is **no** topological obstruction for a metric with bounded geometry of any order.

Bounded maps $C^{\infty,m}(M,N)$:
Consider now $(M^n,g),(N^{n'},h)$ open, complete Riemannian manifolds satisfying (I) and (B_k) and
$f \in C^\infty(M,N)$. Then the differential $f_* = Tf$ is a section of $T^*M \otimes f^*TN$. We endow f^*TN with the induced connection $f^*\nabla^h$. Then ∇^g and $f^*\nabla^h$ induce connections ∇ in all tensor bundles $T^q_s(M) \otimes f^*T^u_v(N)$. Therefore $\nabla^m df$ is well defined. Assume $m \leq k$. We denote by $C^{\infty,m}(M,N)$ the set of all $f \in C^\infty(M,N)$ satisfying

$$ {}^{b,m}|df| := \sum_{i=0}^{m-1} \sup_{x \in M} |\nabla^i df|_x < \infty. $$

Equivalently:
$f \in C^{\infty,m}(M,N) \Leftrightarrow \frac{\partial^\alpha}{\partial x^\alpha} f^\nu$ is uniformly bounded in any normal coordinate system; $|\alpha| \leq m, 1 \leq m \leq k$.

What is the topology and geometry of $C^{\infty,m}(M,N)$?

Review COMPACT case:

Local coordinates in $C^\infty(M,M)$

As for finite dimensional manifolds the parameter space at a point $f \in C^\infty(M,M)$ is isomorphic to the tangent space $T_f C^\infty(M,M) \simeq C^\infty(f^*TM)$ the space of vector fields along f.

Canonical chart at $e = id \in C^\infty(M,M)$**:** $T_e C^\infty(M,M) \simeq Vec^\infty(TM)$ the space of vector field on M. The exponential map $exp_x : T_xM \to M$, $exp_x(v) :=$ geodesic tangent to v through x at $t = 1$. The map $Exp : TM \to M \times M$, $Exp(v_x) := (x, exp_x(v))$ is a local diffeomorphism from a neighborhood $U(0) \subset TM$ onto an open neighborhood $V(\Delta) \subset M \times M$, ($\Delta$ the diagonal in $M \times M$).

Definition:

- $f \in C^\infty(M,M)$ is close to $id \in C^\infty(M,M)$
$\Leftrightarrow graph(f) \subset V(\Delta)$
- $\xi \in Vec^\infty(M)$ is close to $0 \in Vec^\infty(M)$
$\Leftrightarrow \xi(M) \subset U(0)$
- $\Phi(f) := Exp^{-1} \circ (id, f)$ is a local bijection

This defines a chart at id of $C^\infty(M,M)$ if M is **compact**!

We can choose different topologies on the parameter space $Vec(M) \simeq T_eC^\infty(M,M)$:

- C^k - top: $Vec^k(M)$ is a Banach space, $k < \infty$
- C^∞ - top: $Vec^\infty(M)$ is a Fréchet space,
- H^s - top: $Vec^s(M)$ is a Hilbert space, $s \geq \frac{1}{2}dimM$

independent of trivialisation if M is compact.

Facts:

- $\{X\ close\ to\ 0\} \subset Vec^\bullet(M)$ is open $\Leftrightarrow$ M is compact.
- The change of coordinates is C^∞ if M is compact

(α -lemma, Ω - lemma).

Theorem: If M is compact then
$C^k(M,M)$, $C^\infty(M,M)$, $C^s(M,M)$ are smooth Banach- Fréchet-, Hilbert- manifolds.
$Diff^\bullet(M)$ is open in $C^\bullet(M,M)$ hence a smooth Banach- Fréchet-, Hilbert- manifolds.

NON compace case

Let (M^n, g), $(N^{n'}, h)$ be open, complete Riemannian manifolds satisfying the conditions (I), (B_k).
Let $f \in C^{\infty,m}(M,N)$ and $\xi \in C^{\infty}(f^*TN)$.
Define $g_\xi : M \to N \ : \ g_\xi(x) := exp_{f(x)}(\xi(f(x)), \ (\equiv exp\, \xi)$.

Proposition: Assume $m \le k$, $f \in C^{\infty,m}(M,N)$ and $\xi \in C^{\infty}(f^*TM)$ with

$$^{b,m}\,|\xi| := \sum_{i=0}^{m} \sup_{x \in M} |\nabla^i \xi|_x < \delta_N < r_{inj}(N).$$

Then $g_\xi = exp\ \xi \in C^{\infty,m}(M,N)$.

Definition: We define "f close to g" in $C^{\infty,m}(M,N)$ in the L_p-category: Let $0 < \delta < \frac{1}{2} r_{inj}(N), 1 < p < \infty$.

$$V_\delta := \{f, g \in C^{\infty,m}(M,N) \mid \exists \xi \in C^{\infty}(f^*TN) \ s.t. \ g = g_\xi \ \}$$

$$and \ |\xi|_{p,m} := (\int_M \sum_{i=0}^{m} |\nabla^i \xi|_x^p \ d \ vol_x(g))^{1/p} < \delta\}.$$

Theorem (J.Eichhorn): $\mathcal{V} := \{V_\delta\}_{0<\delta<r_{inj}(M)/2}$ is a basis for a metrizable uniform structure on $C^{\infty,m}(M,N)$.

We denote by $\Omega^{p,m}(M,N)$ the completion of $C^{\infty,m}(M,N)$ with respect to this uniform structure. Explicit description of $\Omega^{p,m}(M,N)$:

Proposition: $f \in \Omega^{p,m}(M,N) \Leftrightarrow f = exp_g \xi \circ g$,
$f \in C^{\infty,m}(M,N),\ \xi \in C^{\infty}(g^*TN, g^*\nabla),\ |\xi|_{p,m} < \varepsilon$.

Theorem: Let $(M^n, g), (N^{n'}, h)$ be open, complete Riemannian manifolds of bounded geometry of order $k,\ 1 < p < \infty,\ m \le k,\ m > \frac{n}{p} + 1$. Then each component of $\Omega^{p,m}(M,N)$ is a C^{k+1-m}-Banach manifold, and for $p = 2$ it is a Hilbert manifold.

Remarks:

- Neighborhood of $f \in \Omega^{p,m}(M,N)$: let $0 < \varepsilon < r_{inj}(N)$

$\mathcal{U}_\varepsilon = \{g \in \Omega^{p,m}(M,N) | g = exp\,\xi,\ \xi \in C^{\infty}(f^*TN),\ |\xi|_{p,m} < \varepsilon$

- $T_f\Omega^{p,m}(M,N) = \Omega^{p,m}(f^*TN)$
- Change of coordinates: $exp_g \circ exp_f^{-1}$ is C^{k-m+1}.

If M is compact then $k = \infty$.

- If $g \in comp(f)$ then $T_f\Omega^{p,m}(M,N) \simeq T_g\Omega^{p,m}(M,N)$ otherwise not.

The bounded diffeomorphism group $Diff^{p,m}(M)$.

Problem: $C^{\infty,m}(M)\bigcap Diff(M)$ is **not** a group, i.e $f\in C^{\infty,m}(M)\bigcap Diff(M)\not\Rightarrow f^{-1}\in C^{\infty,m}(M)$.
We need an additional assumption :
Let (M^n,g) be an oriented, open, complete Riemannian manifold of bounded geometry of order k, $1<p<\infty$, $k\geq m>\frac{n}{p}+1$. A choice of an orthonormal basis in each T_xM implies that $|\lambda|_{min}(df)$, the absolute value of the eigenvalues of the Jacobian of f, is well defined. Set

$$Diff^{p,m}(M):=\{f\in\Omega^{p,m}(M,M)\mid f\; is\; injective,\; subjective$$

$$preserves\; orientation\; and\; |\lambda|_{min}(df)>0\}.$$

Theorem: $Diff^{p,m}$ is open in $\Omega^{p,m}(M,M)$ in particular, each component is a C^{k+1-m}-Banach manifold, and for $p=2$ it is a Hilbert manifold.

Theorem; Assume $(M^n,g),k,p,m$ as above.
a) Assume $f,g\in Diff^{p,m}, g\in\ comp\ (id_M)\subset Diff^{p,m}$.
Then $g\circ f\in Diff^{p,m}$ and $g\circ f\in\ comp\ (f)$.
b) Assume $f\in\ comp(id_M)\subset Diff^{p,m}$.
Then $f^{-1}\in\ comp(id)\subset Diff^{p,m}$.
c) $comp(id)$ is a metrizable topological group.

Differentiability of the group operations

The differentiability of group multiplication (composiyion) and inversion follows from the α - and ω - lemma.

Theorem: (α - lemma)
Assume $m \leq k,\ m > \frac{n}{p}+1,\ f \in Diff^{p,m}(M)$.
Then the right multiplication

$$\alpha_f : Diff_0^{p,m}(M) \to Diff^{p,m}(M),\ \alpha_f(g) = g \circ f$$

is of class C^{k+1-m}.

Theorem: (ω - lemma)
Let $k+1-(m+s) > s,\ f \in Diff_0^{p,m+s}(M) \subset Diff_0^{p,m}(M),\ m > \frac{n}{p}+1$.
Then the left multiplication

$$\omega_f : Diff^{p,m}(M) \to Diff^{p,m}(M),\ \omega_f(g) = f \circ g$$

is of class C^s.

Main Theorem:(J. Eichhorn)
Let (M^n, g) be an open, oriented, complete Riemannian manifold satisfying (I), (B_∞) . Let $Diff_0^{p,\infty}(M) :=$ $\lim_{\leftarrow} Diff_0^{p,m}(M)$ with the inverse limit toplogy. Then

$$\{Diff_0^{p,\infty}(M), Diff_0^{p,m}(M) \mid m > \frac{n}{p}+1\}$$

is an ILB - Lie group; and for $p = 2$ it is an ILH - Lie group.

Volume preserving and symplectic diffeomprphisms

Theorem: Assume (M^n, g) is an open manifold satisfying the conditions (I) and $(B_k), k \geq m \geq r > \frac{n}{2} + 1$ and the spectral condition inf $\sigma_{ess}(\Delta_1|_{(ker \Delta_1)^\perp}) > 0$. Let ω be a C^m-bounded closed q-form with $\inf_{x \in M} |\omega|_x^2 > 0$, and consider $D_\omega^r = \{f \in D^r | f^* \omega = \omega)\}$. Then the group $D_{\omega,o}^r = D_o^r \cap D_\omega^r$ is a C^{k-r+1} submanifold of D_o^r.

Theorem: Assume (M^n, g) with (I) and (B_∞) . Let ω be a C^∞-bounded strongly nondegenerate q-form, $q = n$ or $q = 2$, and assume the spectral condition above. Set $D_{\omega,o}^\infty = \lim_{\leftarrow r} D_{\omega,o}^r$. Then $\{D_{\omega,o}^\infty \,, D_{\omega,o}^r | \; r > \frac{n}{2} + 1\}$ is an ILH-Lie group and the Lie algebra of $D_{\omega,o}^\infty$ consists of divergence free ($q = n$), or locally Hamiltonian $(q = 2)$ vector fields X with finite Sobolev norm $|X|_r$ for all r.

Applications

1) Euler‘s equations of incompressible fluids are equivalent to geodesics on $Diff_{\mu,0}^{\infty,m}$.
Analogous results hold for compressible fluids (in progress).

2) The group of bounded quantomorphisms on an open manifold form an ILH - Lie group.

3) The group of invertible , bounded pseudodifferential operators form an ILH - Lie group.

4) The group of invertible , bounded Fourier integral operators form an ILH - Lie group.