BRST BICOMPLEXES and COHOMOLOGIES

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ICM 94, International Congress of Mathematicians, Zuerich Switzerland, 1994

BRST bicomplexes and cohomologies appear in different forms and fashions in the mathematics and physics literature. We discuss a few of them related to BRST symmetries and anomalies:

1. Variational bicomplex
2. BRST bicomplex (Chevalley - Eilenberg)
3. Faddeev bicomplex
4. Koszul-Tate bicomplex (quantum BRST)
5. Weil bicomplex
6. semi infinite cohomology (quantum BRST)
7. Gelfand-Fuks complex
8. Cech-DeRham complex
9. foliations (spectral sequences)
10. Lie group/algebra- cohomology

1. Variational Bicomplex

(F. Takens, I. Anderson: calculus of variation, inverse problem)

Consider a fiber bundle $\pi : P \to M^n$ **, coordinates** $(x, u) \mapsto x$

associated jet bundle $J^\infty(\pi)$ **, coord.** $(x, u, u_x, u_{xx}$

the space $\Omega^p(J^\infty(\pi))$ **of p - forms on** $J^\infty(\pi)$ **is bigraded**

horizontal^r**, and vertical**^{s}

$$\Omega^p(J^\infty(\pi)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(\pi))$$

which induces d_H **and** d_V **derivatives**

$$d = d_H + d_V : \Omega^{r,s} \to \Omega^{r+1,s} \oplus \Omega^{r,s+1}.$$

Poincare lemmas hold for d_H **,** d_V **and we have** $d_H^2 = 0$**,** $d_V^2 = 0$**,** $d_H d_V = -d_V d_H$**, hence** $d^2 = 0$**.**

(augmented) Variational Bicomplex

$$
\begin{array}{ccccccccccc}
 & & \uparrow d_V & & & & & & \uparrow d_V & & \\
0 & \longrightarrow & \Omega^{0,3} & & & \cdots & & & \Omega^{n,3} & \xrightarrow{I} & \\
 & & \uparrow d_V & & & & & & \uparrow d_V & & \\
0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} \cdots & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} & \xrightarrow{I} & \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \\
0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} \cdots & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} & \xrightarrow{I} & \\
 & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & \\
\mathrm{IR} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} \cdots & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0} & &
\end{array}
$$

$\mathcal{F}^s = \{\omega \in \Omega^{n,s} | I(\omega) = \omega\}$ **functional forms,**

I **Euler operator** $(I^2 = 0,\ I \circ d_H = 0)$

$d = d_H + d_V : \Omega^p \to \Omega^{p+1}$ **defines <u>local</u> cohomologies**

$H^p(\pi) = ker\, d/im\, d$ **which can be computed explicitly by spectral sequences.**

These are <u>local</u> cohomologies because for any $\omega \in \Omega^{r,s}(J^\infty(\pi))$**,** $j \in J^\infty(\pi)$ **and vector fields** $X_1, \ldots, X_s$ **on** $J^\infty(\pi)$ **we have the local form** $\omega(j, X_1, \cdots, X_s) \in \Omega^r(M)$ **defined by**

$$\omega(j, X_1, \cdots, X_s)(x) = (i_{X_1(j)} \cdots i_{X_s(j)}\omega)(x, j)$$

which is a local r - form on M**, i.e. depending only on finite derivatives (jets) at** x**.**

2. BRST Bicomplex

(R.Schmid, L. Bonora, P. Cotta-Ramusino, J.A. Dixon, ...)

Consider a principal $G-$ **bundle** $\pi : P \to M$

$\Omega^p(P, LieG)$ = **Lie algebra** ($LieG$) **valued equivariant p-forms**

$\mathcal{G}$ = **Lie group of gauge transformations with Lie algebra** $Lie\mathcal{G}$

Set $\mathbf{C}^{q,p}_{loc} = \Lambda^q(Lie\mathcal{G}, \Omega^{p,0}(J^\infty(\pi))$ **local** q **- cochains**

$\delta_{loc} : \mathbf{C}^{q,p}_{loc} \to \mathbf{C}^{q+1,p}_{loc}$ **Chevalley-Eilenberg coboundary operator with resp. to representation** ρ **of** $Lie\mathcal{G}$ **on** $\Omega^p(P, LieG)$

$$\begin{aligned}(\delta_{loc}\phi)(\xi_0, \cdots, \xi_q) \ = \ & \sum_{i=0}^{q}(-1)^i \rho(\xi_i)\phi(\xi_0, \cdots, \hat{\xi}_i, \cdots, \xi_q) \\ & + \sum_{i<j}(-i)^{i+j}\phi(\rho(\xi_i)\xi_j, \cdots, \hat{\xi}_i, \cdots, \xi\end{aligned}$$

$\Rightarrow$ **nilpotency** $\delta^2_{loc} = 0$ **.**

Define the BRST operator $\mathbf{s} : \mathbf{C}^{q,p}_{loc} \to \mathbf{C}^{q+1,p}_{loc}$

$$\mathbf{s} \equiv \frac{(-1)^{p+1}}{q+1}\delta_{loc}$$

$\Rightarrow$ **nilpotency** $\mathbf{s}^2 = 0$

<u>**Theorem:**</u>

For any vector potential $A \in \mathbf{C}^{0,1}_{loc} = \Lambda^0(Lie\mathcal{G}, \Omega^1$ **ghost field** $\eta \in \mathbf{C}^{1,0}_{loc} = \Lambda^1(Lie\mathcal{G}, \Omega^{0,0})$ **the Maurer** **form on** $\mathcal{G}$**, i.e.** $\eta = id : Lie\mathcal{G} \to Lie\mathcal{G}$ **, the classi** **BRST transformations are**

$$\mathbf{s}A = d\eta + [A, \eta] \ , \qquad \mathbf{s}\eta = -\tfrac{1}{2}[\eta, \eta] \ , \qquad \mathbf{s}\bar{\eta} = b \ ,$$

3. Faddeev Bicomplex

(J. Stasheff, B. Zumino, de Rham bar construction)

Consider a principal $G-$ **bundle** $\pi : P \to M$**, and**

$G^p = \{f : S^p \to G | \infty \to 1\}$ **the space of** p**-loops. We have**

the exterior derivative $d : \Omega^q(P \times G^p) \to \Omega^{q+1}(P \times G^p)$

and the simplicial group coboundary operator

$\Delta : \Omega^{q-p}(P \times G^p) \to \Omega^{q-p}(P \times G^{p+1})$ $\Delta = \Sigma(-1)^i \Delta_1$ **where**

$\Delta_i : P \times G^{p+1} \to P \times G^p : (x, g_1, \cdots, g_{p+1}) \mapsto (x, g_1, \cdots, g_i g_{i+1}, \cdots, g_{p+1})$.

We have $d^2 = 0$**,** $\Delta^2 = 0$ **and** $d\Delta + (-1)^p \Delta d = 0$ **hence the**

total derivative $D = \Delta + (-1)^p d$ **gives** $D^2 = 0$**.**

(De Rham bar construction)

Example: $S^p = S^3$ **the Chern-Simon form** $\omega_5^0 = Tp(A)$, **where** $dTp(A) = p(F) = Tr\ F^3$ ($p =$ **invariant polynomial,** $T =$ **transgression)**

We get the stair case equations:

$$
\begin{array}{c|cccccccc}
\mathbf{q=3} & \mathbf{Tr}\ F^3 & & & & & & \\
 & \uparrow \mathbf{d} & & & & & & \\
\mathbf{q=2} & \omega_5^0 & \xrightarrow{\Delta} & \bullet & & & & \\
 & & & \uparrow \mathbf{d} & & & & \\
\mathbf{q=1} & & & \omega_4^1 & \xrightarrow{\Delta} & \bullet & & \\
 & & & & & \uparrow \mathbf{d} & & \\
\mathbf{q=0} & & & & & \omega_3^2 & \xrightarrow{\Delta} & \mathbf{0} \\
\hline
 & \mathbf{p=0} & & \mathbf{p=1} & & \mathbf{p=2} & & \mathbf{p=3}
\end{array}
$$

ω_3^2 **represents the anomaly.**

4. Koszul-Tate Bicomplex

(B. Kostant, S. Sternberg, M. Henneaux)

Consider a Poisson manifold $(M, \{\ ,\ \})$ **with Hamiltonian**

G **action. Extend the momentum map** $J : LieG \to C^{\infty}(M)$

to a super derivative δ **, extend the Lie algebra** d

$d : \Lambda LieG \otimes C^{\infty}(M) \to Lie^{*}G \otimes (\Lambda LieG \otimes C^{\infty}(M))$

to $\tilde{d}$

such that we have

$$\begin{array}{ccc} \Lambda^{p} Lie^{*}G \otimes \Lambda^{q} LieG \otimes C^{\infty}(M) & \xrightarrow{\delta} & \Lambda^{p} Lie^{*}G \otimes \Lambda^{q} \\ \tilde{d} \downarrow & & \\ \Lambda^{p+1} Lie^{*}G \otimes \Lambda^{q} LieG \otimes C^{\infty}(M) & & \delta \ and \ \tilde{d} \ \ \textbf{be} \end{array}$$

$$\delta(\omega \otimes \xi \otimes 1) = \omega \otimes 1 \otimes J(\xi)\ ,\ \delta(\omega \otimes 1 \otimes f) = \omega \otimes 0\ ,\ \tilde{d}(\omega \otimes k$$

We have $\delta^{2} = 0, \tilde{d}^{2} = 0$ **and** $\tilde{d}\delta - \delta\tilde{d} = 0$ **.**

The total differential defines the BRST operator

$$D = \tilde{d} + (-1)^p 2\delta \;:\; \mathcal{C}^k \to \mathcal{C}^{k+1}, \quad \textbf{where}$$

$$\mathcal{C}^k = \Sigma_{p-q=k}\ \Lambda^p Lie^* G \otimes \Lambda^q LieG \otimes C^\infty(M)$$

$\Rightarrow$ **nilpotency** $D^2 = 0$

<u>**Theorem**</u> **: (Kostant, Sternberg)**
Functions on the reduced phace space are giv
the cohomology
$C^\infty(J^{-1}(0)/G) = H_D(\Lambda Lie^* G \otimes \Lambda LieG \otimes C^\infty(M))$
$E_2^{0,0}$ **of the associated spectral sequence.**

Quantize D **on** $Cliff(LieG \oplus Lie^*G)$ **to get quantized BRST operator** $Q = \frac{1}{4} C^k_{ij}(\alpha^i\alpha^j\xi_k + \alpha^i\xi_k\alpha^j + \xi_k\alpha^i\alpha^j) \Rightarrow Q^2 = 0$.

5. Weil Bicomplex

Consider a principal $G-$ **bundle** $\pi : P \to M$**, and** $\Lambda(Lie^*\mathcal{G})$ **the exterior algebra ,** $\mathbf{S}(Lie^*\mathcal{G})$ **the symmetric algebra of** $Lie\mathcal{G}$**. The Weil algebra**

$$W(Lie\mathcal{G}) \equiv \Lambda(Lie^*\mathcal{G}) \otimes \mathbf{S}(Lie^*\mathcal{G})$$

is a graded differential $\mathcal{G}$ **algebra**

$W(Lie\mathcal{G}) = \Sigma_r W^r$, $W^r = \Sigma_{p+2q=r}\Lambda^p(Lie^*\mathcal{G})\otimes S^q(Li$

Antiderivation δ_W **on** $W(Lie\mathcal{G})$**,** $\delta_W = \delta_\Lambda + \delta_S + h$ **, where**

$$\delta_\Lambda : W^p \to W^{p+1}$$

$(\delta_\Lambda\phi)(x_0, \cdots, x_p) = \sum_{\nu<\mu}(-1)^{\nu+\mu}\phi([x_\nu, x_\mu], x_0, \cdots, \hat{x}_\nu,$

$\phi \in \Lambda^p(Lie\mathcal{G})$, $x_i \in Lie\mathcal{G}$**. With** $\{e_\alpha\}$ **basis of** $Lie\mathcal{G}$**,** $\{\theta^\alpha\}$ **dual basis of** $Lie^*\mathcal{G}$**,** L_Λ **= Lie derivative ,** $\mu(a)b \equiv a \wedge b$**, we have**

$\delta_\Lambda = \frac{1}{2}\Sigma_\alpha \mu(\theta^\alpha) L_\Lambda(e_\alpha)$ **. Nilpotency** $\delta_\Lambda^2 = 0$.

$$\delta_S = \Sigma_\alpha \mu(\theta^\alpha) L_S(e_\alpha) : W^p \to W^{p+1}$$

Note $\delta_s^2 \neq 0$ **but** $(\delta_\Lambda + \delta_S)^2 = 0$, **so** $\delta_S^2 = -(\delta_\Lambda \delta_S + \delta_S \delta_\Lambda)$**.**

$$h = \Sigma_\alpha \mu(\theta^\alpha) \otimes i_A(e_\alpha) : W^p \to W^{p+1}$$

(i_A **= interior product).**

The BRST operator is total differential $\delta_W = \delta_\Lambda + \delta_S + h$**.**

The associated anomalies in $H^1(Lie\mathcal{G})$ **can be computed**

explicitly.

6. Wess-Zumino Consistency Condition for BRST

This is a problem in local cohomology BRST bicomplex $\mathcal{C}^*_{loc} = \{\mathcal{C}^{q,p}_{loc}, \Delta\}_{q,p\in\mathbf{N}}$ with total differential

$$\Delta = \delta_{loc} + (-1)^p d$$

$\delta_{loc} : \mathcal{C}^{q,p}_{loc} \longrightarrow \mathcal{C}^{q+1,p}_{loc}$, **Chevalley-Eilenberg coboundary**

$d : \mathcal{C}^{q,p}_{loc} \longrightarrow \mathcal{C}^{q,p+1}_{loc}$ **exterior derivative**

$$\Delta^2 = \delta_{loc}d + d\delta_{loc} = \delta^2_{loc} = d^2 = 0$$

Wess-Zumino consistency condition for $\omega \in \mathcal{C}^*_{loc}$ **means there exists** $\alpha \in \mathcal{C}^*_{loc}$ **such that**

$$\delta_{loc}\omega + d\alpha = 0 \qquad (WZ)$$

Any solution of (WZ) of the form

$$\omega = \delta_{loc}\beta + d\gamma, \;\; \beta, \gamma \in \mathcal{C}^*_{loc}$$

is trivial, $\delta_{loc}\omega = 0$**.**

Consistency condition (WZ) produces descent equations as follows: If $\delta_{loc}\omega + d\alpha = 0$ **taking** δ_{loc} **of (WZ) we get**

$\delta_{loc}{}^2\omega + \delta_{loc}d\alpha = 0$ **hence** $\delta_{loc}d\alpha = 0$**.**

Poincare lemma: exists local form β **s.t.** $\delta_{loc}\alpha + d\beta = 0$**.**

By definition $\delta_{loc}[\omega] = [\alpha]$**.**

If ω **is trivial, i.e.** $\omega = \delta_{loc}\beta + d\gamma$ **then** $\delta_{loc}d\gamma = -d\alpha$**,**

hence $\alpha = \delta_{loc}\gamma + d\lambda$**, i.e.** $[\alpha] = 0$**.**

We get the descent equations

$$\begin{aligned}\delta_{loc}\omega + d\omega_1 &= 0\\ \delta_{loc}\omega_1 + d\omega_2 &= 0\\ &\cdot\\ &\cdot\\ &\cdot\\ \delta_{loc}\omega_{k-1} + d\omega_k &= 0\end{aligned}$$

where k **is the smallest integer such that** $[\omega] \in \mathbf{H}^k_{loc}(Lie\mathcal{G})$ **with** $\delta_{loc}\omega = 0$**.**

7. g - Symplectic Structures on Orbits

g $=$ **Lie algebra** $LieG$**. A g-symplectic structures on** P **is a closed, nondegenerate g - form** $\Omega \in \Omega^2(P, \mathbf{g})$ **, i.e.** $d\Omega = 0$ **and for each** $p \in P$ **induced linear injective map**

$$\Omega(p)^{\#} : T_pP \to L(T_pP, \mathbf{g})\ \Omega(p)^{\#}(v) \cdot w = \Omega(p)(v, w).$$

A vector field X **on** P **is called g - Hamiltonian if exists g-function** $f : P \to \mathbf{g}$ **such that** $df = i_X\Omega$**.**

A g - vector field X **is locally g - Hamiltonian iff its flow** φ_t **is g - symplectic, i.e** $\varphi_t^*\Omega = \Omega$**.**

Poincare: For any $\alpha \in \Omega^p(\mathbf{R}^n, \mathbf{g})$, $d\alpha = 0$ exists locally

$\beta \in \Omega^{p-1}(\mathbf{R}^n, \mathbf{g})$ **s.t.** $\alpha = d\beta$ **and**

$$\beta(x) = \int_0^1 i_x \alpha(tx) dt \ .$$

<u>Theorem:</u>
If G is semi simple, then every G - orbit $\mathcal{O}_p$ o
action R_g of G on P is a g - symplectic manif
by the Maurer Cartan form θ on G.
The symplectic form ω_p on $\mathcal{O}_p$ is given by

$$\omega_p = dR_p^* \theta$$

8. Canonical Momentum Map on $\mathcal{O}_p$

Proposition:

For every $\xi \in$ g the fundamental vector field ξ_p on $\mathcal{O}_p$ defined by

$$\xi_p(q) = \frac{d}{dt|_{t=0}} R_{exp\ t\xi}(q)$$

is locally g -Hamiltonian.

Corollary:

For every $\xi \in$ g, there exists a g- function $H : \mathcal{O}_p \to$ g such that $\xi_p = X_H$ i.e. $dH = i_{\xi_P} \Omega_p$. Explicitly

$$H(x) = -\frac{1}{2}[x, x \cdot \xi].$$

Proposition:

The g -momentum map of the right action

of G **on** $\mathcal{O}_p$**,** $J : \mathcal{O}_p \to \mathcal{L}(\mathbf{g}, \mathbf{g})$ **defined by** $\langle J(q), \xi \rangle = H(q)$, $q \in \mathcal{O}_p$, $\xi \in \mathbf{g}$ **is given by**

$$J(q) = ad_\eta \circ TR_q \ ,$$

where $\eta = R_p^* X_t(g)$, $q = p \cdot g$ **.**

9. Solution of Wess - Zumino Consistency Condition

Apply the results above to infinite dimensional analogue. Consider the principal $\mathcal{G}$ bundle $(\mathcal{P}, \pi, \mathcal{M})$, where $\mathcal{P} = \Omega^*(P, \mathbf{g})$ under adjoint $\mathcal{G}$ action and $\mathcal{M} = \mathcal{P}/\mathcal{G}$.
For $A \in \Omega^*(P, \mathbf{g})$ the canonical 1-form Θ_A on the $\mathcal{G}$ - orbit $\mathcal{O}_A$ induced from the Maurer - Cartan form Θ on $\mathcal{G}$ becomes a map

$$\Theta_A : \mathcal{O}_A \to \Omega^1(P, \mathbf{g}) \simeq \mathcal{C}^{0,1}_{loc}$$

and the momentum map becomes

$$J : \mathcal{O}_A \to \mathcal{L}(Lie\mathcal{G}, Lie\mathcal{G}) = \mathcal{C}^{1,0}_{loc}$$

<u>Theorem</u>:

The momentum map J satisfies the consisten
for the canonical Maurer-Cartan form Θ of $\mathcal{G}$

$$\delta_{loc}\Theta_A + dJ = 0.$$