Geometries, the principle of duality, and algebraic groups[*]

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Abstract

J. Tits gave a general recipe for producing an abstract geometry from a semisimple algebraic group. This expository paper describes a uniform method for giving a concrete realization of Tits's geometry and works through several examples. We also give a criterion for recognizing the automorphism of the geometry induced by an automorphism of the group. The E_6 geometry is studied in depth.

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J. Tits's theory of buildings associated with semisimple algebraic groups gives a unified method of extracting a geometry from a group. For example, the group SL_n gives rise to $(n-1)$-dimensional projective space. Tits's geometry however is very abstract. Speaking precisely, one obtains an *incidence geometry*, which consists of an abstract set of objects each with a given type, and a reflexive, symmetric binary relation on the set of objects called incidence. We find it more palatable to think of projective space in a concrete way, as the collection of subspaces of some explicit vector space. In Section 2 we give an explicit recipe for concretizing Tits's incidence geometry.

The midsection of this paper consists of explicit descriptions of the concrete realizations of the geometries for split groups of type A, D, E_6, E_7, F_4, and G_2. (Readers should have little trouble filling in the missing types B and C. We do not know a good description for the geometry of type E_8, but we make a few comments in 9.3.) Such descriptions may be found in a variety of places in the literature, e.g., [Coh] or [FF]. The main innovation here is that our recipe produces a realization of the geometry by a largely deterministic process beginning from the root system of the group and a fundamental representation, whereas approaches in the literature have the appearance of being ad hoc. Roughly speaking, Tits developed his theory of buildings by abstracting and unifying known properties of the various concrete geometries [T 74, p. v], so our approach here reverses the historical development.

Our principal tool is the representation theory of semisimple groups; we only use the most elementary results, but we exploit those ruthlessly. Consequently, throughout this paper, *our base field k is assumed to have characteristic zero.* Some results hold over an arbitrary field, see Remarks 5.5 and 13.22.

The final portion of this paper concerns duality. Élie Cartan was already aware (see [Ca 25, p. 362]) that the "outer" automorphism $g \mapsto (g^{-1})^t$ (where t denotes transposition) of SL_3 gives rise to a polarity in the projective plane and so to the principle of duality. In general, an outer automorphism of a semisimple group gives an automorphism ψ of the corresponding geometry, hence also a "principle of duality" (or "triality" or ...). In Sections 10–13 below, we give a criterion for recognizing such an automorphism ψ and apply our criterion to get an explicit description of ψ in essentially all cases. The explicit description of ψ for the E_6 geometry is new. Having an explicit formula for ψ is useful for giving a concrete description of the projective homogeneous varieties for groups of "outer type", see for example [MPW, p. 172].

We hope that readers who are not familiar with, say, exceptional groups will find the presentation here unusually accessible because of the uniform treatment in the common language of representation theory. Experts will note that the concept of inner ideal occurs naturally in the E_7 geometry (§8) and that we do not need to mention octonions at all in our discussion of triality in §12.

Our hypothetical reader is moderately familiar with the theory of linear algebraic groups as in [Bor], [Sp 98], or [St lect] and the classification of irreducible modules via highest weight vectors from [Hu 80, Ch. VI], [FH, §14], [GW, §5.1], or [Va].

1 Tits's geometry Γ_P

In this section, we describe Tits's recipe for producing an incidence geometry from a certain kind of algebraic group. An *incidence geometry* is a set of objects, each of some type (e.g., point, line), together with a symmetric binary relation known as *incidence.* There is just one further axiom: objects of the same type are incident if and only if they are equal.

Remark. Modern formulations of Tits's recipe take a group and construct a building rather than an incidence geometry. From our perspective, a building is an incidence geometry with extra structure that we do not really need. So we deal only with the much-simpler-to-define incidence geometries as in [T 56]. For a presentation in terms of buildings, see [T 74], [TW, 42.3.6], [Br build, Ch. V], or [Sc, §4.4].

We start with a root system Φ in the sense of [St lect, §1], a "reduced root system" in the language of [Bou Lie]. There is a "split" simply connected algebraic group G with root system Φ, and it is uniquely determined up to isomorphism. Taking Φ of type A_n, we obtain a group G isomorphic to SL_{n+1}. When k is algebraically closed, every semisimple algebraic group is obtained in this fashion, or is a quotient of a group obtained in this fashion. For example, the group $SO_{2n}(\mathbb{C})$ is a quotient of $\mathrm{Spin}_{2n}(\mathbb{C})$, which is constructed from the root system D_n.

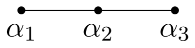
1.1. Parabolic subgroups. A *Borel subgroup* is a maximal closed, connected, solvable subgroup of G. Fix one and call it B; it (combined with a split maximal torus T contained in it) determines a set of simple roots Δ in Φ. We abuse notation by writing Δ also for the associated Dynkin diagram.

A closed subgroup of G is called *parabolic* if it contains a Borel subgroup, and we call a parabolic subgroup *standard* if it contains the Borel B. There is an inclusion-reversing bijection

$$\begin{array}{rcl} \boxed{\text{standard parabolic subgroups of } G} & \leftrightarrow & \boxed{\text{subsets of } \Delta} \\ B & \leftrightarrow & \Delta \\ G & \leftrightarrow & \emptyset \end{array}$$

The maximal proper standard parabolics are in one-to-one correspondence with the elements of Δ. We write P_δ for the standard parabolic corresponding to $\delta \in \Delta$.

1.2 Example (Parabolics in SL_4). As an illustration, the Dynkin diagram of SL_4 is



Here we have labeled the vertices as in [Bou Lie]. The upper triangular matrices are a Borel subgroup, which we take to be B. For our maximal torus T, we take the diagonal matrices. The maximal proper standard parabolics have the form

$$P_{\alpha_1} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \quad P_{\alpha_2} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \quad P_{\alpha_3} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}. \tag{1.3}$$

1.4. Tits's geometry Γ_P ([T 74, §5], [TW, 42.3.6]). Tits defines the objects of the incidence geometry to be the maximal proper parabolic subgroups of G. Since the Borel subgroups of G are all conjugate, every parabolic is conjugate to a unique standard parabolic [Bor, 21.12]. Therefore, every maximal proper parabolic subgroup corresponds to a unique element $\delta \in \Delta$; this is the *type* of the parabolic. Two maximal proper parabolics are said to be *incident* if their intersection contains a parabolic subgroup. We write Γ_P for this incidence geometry, where the subscript P is meant to remind the reader that the objects are parabolic subgroups of G.

In the case of SL_4, one typically calls the parabolics of type α_1 "points", α_2 "lines", and α_3 "planes". This identifies the geometry Γ_P with 3-dimensional projective space.

Note that the notion of incidence we defined for Γ_P satisfies the axiom from the beginning of the section: If parabolics P and P' of the same type are incident, then $P \cap P'$ contains a Borel subgroup, which after conjugation we may assume is the standard Borel B. But by hypothesis, P and P' have the same type, hence $P = P'$.

2 A concrete geometry Γ_V, part I

We now take Tits's incidence geometry Γ_P—whose objects are certain parabolic subgroups of G—and produce an isomorphic incidence geometry Γ_V whose objects are subspaces of a fixed vector space V. For example, in the case where G is SL_n, Γ_V consists of the nonzero, proper subspaces of k^n.

Fix a representation of G on a finite-dimensional vector space V, i.e., a homomorphism of algebraic groups $G \to GL(V)$. We assume that the representation is nontrivial (i.e., the image of G is not just the identity transformation) and irreducible (i.e., there is no nonzero, proper G-invariant subspace). For each vertex δ of the Dynkin diagram, we choose a nonzero, proper subspace V_δ of V that is invariant under the parabolic P_δ.[1] Every maximal proper parabolic subgroup P' is conjugate to a unique standard parabolic subgroup P_δ, and we define a subspace $V_{P'}$ via

$$V_{P'} := gV_\delta \quad \text{for } g \in G(k) \text{ such that } P' = gP_\delta g^{-1}.$$

Of course, g is not uniquely determined, but if $h \in G(k)$ also satisfies $hP_\delta h^{-1} = P'$, then $h^{-1}g$ normalizes P_δ. Since P_δ is its own normalizer [Bor, 11.16], g equals hp for some $p \in P_\delta(k)$ and $hV_\delta = gV_\delta$. We remark that the stabilizer of $V_{P'}$ in G is precisely P'. Indeed, the stabilizer of $V_{P'}$

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[1] For the moment, we assume that such a subspace exists. The doubting reader may wish to glance ahead at Prop. 3.4.

is a closed subgroup of G containing P', hence must be P' or G. Since the representation V is irreducible, the stabilizer is P'.

We define an incidence geometry Γ_V whose objects are the subspaces V_P of V, as P ranges over the maximal proper parabolic subgroups of G. The map $P \mapsto V_P$ is surjective by definition, but it is also injective because P is precisely the stabilizer of V_P. Therefore, the sets of objects in Γ_P and Γ_V are isomorphic. Define the notions of type and incidence in Γ_V by transporting them from Γ_P. Speaking precisely, we say that V_P in Γ_V has type $\delta \in \Delta$ if the parabolic P is of type δ. We define two objects in Γ_V to be incident if and only if the corresponding parabolic subgroups are incident.

In this very simple way, we have obtained a realization of Tits's abstract geometry Γ_P as a collection of subspaces of the concrete vector space V. This recipe begs two obvious questions:

Are we guaranteed that a nonzero, proper P_δ-invariant subspace of V exists? (2.1)

Is there a way to tell if two subspaces of V in Γ_V are incident without discussing the corresponding parabolic subgroups? (2.2)

We will address these two questions in the next section and the examples that follow it. But first, here is an example to illustrate the construction.

2.3 Example (Γ_V for SL_4). Referring to the description of the parabolic subgroups of SL_4 from Example 1.2, we see that P_{α_i} stabilizes the i-dimensional subspace of k^4 consisting of vectors whose only nonzero entries are in the first i coordinates. We can take this to be V_{α_i}. The objects of Γ_V are all proper, nonzero subspaces of V.

We claim that two elements of Γ_V are incident if and only if one is contained in the other. Indeed, let W, W' be proper, nonzero subspaces, stabilized by maximal proper parabolics P, P' in SL_4. If W and W' are incident, then $P \cap P'$ contains a Borel subgroup. After conjugation we may assume that P and P' are standard, hence appear in (1.3). Then clearly W is contained in W' or vice versa. Conversely, if W is contained in W', there is some $g \in SL_4(k)$ such that gW, gW' are equal to V_{α_i}, $V_{\alpha_{i'}}$ for some $i \le i'$. Then gPg^{-1} and $gP'g^{-1}$ equal P_{α_i} and $P_{\alpha_{i'}}$ respectively, hence the parabolics P, P' are incident.

3 A concrete geometry Γ_V, part II

We will now make the geometry Γ_V from the previous section more concrete by focusing on the case where V is a fundamental irreducible representation of G (i.e., the highest weight of V is a fundamental weight). We completely answer (2.1) in the affirmative in Prop. 3.4 and we partially answer (2.2) in Prop. 3.7.

We view G as being constructed from the root system Φ by the Chevalley construction as in [St lect, §6]. That is, it is generated by the images of homomorphisms $x_\alpha : \mathbb{G}_a \to G$ as α ranges over the roots in Φ. Write $\mathfrak{X}_\alpha$ for the image of x_α. For each $\alpha \in \Phi$, the map $t \mapsto x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(1-t)x_{-\alpha}(1)x_\alpha(-1)$ defines a homomorphism $\mathbb{G}_m \to G$, which we denote by h_α. The images of the h_α's generate a maximal torus T in G. We fix a set of simple roots Δ in Φ and choose our standard Borel subgroup B to be the one generated by T and the $\mathfrak{X}_\rho$ for ρ a positive root.

Fix a root $\beta \in \Delta$ and let ω be the corresponding fundamental weight. In this section, V denotes the irreducible representation with highest weight ω with respect to our choice of torus T and Borel B. Fix a highest weight vector v^+ in V.

3.1. For a simple root $\delta \in \Delta \setminus \{\beta\}$, we define the *$\delta$-component* of Δ to be the connected component of β in $\Delta \setminus \{\delta\}$. The *$\beta$-component* is the empty set.

Now fix a $\delta \in \Delta$. For each root ρ of G, write $\rho = \sum_{\alpha \in \Delta} c_\alpha \alpha$ for integers c_α; we define the *δ-height* of ρ to be $\sum c_\alpha$ as α ranges over the simple roots *not* in the δ-component. In the case $\delta = \beta$, this is the usual notion of height.

Write L_δ for the subgroup of G generated by the root subgroups $\mathfrak{X}_\rho$ as ρ ranges over the roots of δ-height zero. The description of G in terms of generators and relations shows that L_δ is a simple group whose Dynkin diagram is the δ-component. (It is the semisimple part of the Levi subgroup of the parabolic corresponding to the complement of the δ-component.) We set V_δ to be the subspace $L_\delta\, v^+$ spanned by the L_δ-orbit of v^+. We remark that L_β is the group with one element and V_β is the line kv^+.

3.2 Lemma. *For each $\delta \in \Delta$, the subspace V_δ is a direct sum of the weight spaces in V with weights of the form $\omega - \alpha$ where α has δ-height zero.*

Recall that every weight of V is of the form $\omega - \alpha$ for α a sum of positive roots. For the proof, we will use repeatedly [St lect, p. 209, Lemma 72], which says: For $v \in V$ of weight $\omega - \alpha$ and ρ a root,

$$x_{-\rho}(t)v = v + \sum_{i \geq 1} t^i v_i, \tag{3.3}$$

for some $v_i \in V$ with weight $\omega - (\alpha + i\rho)$.

Proof. Write V as $V_0 \oplus V_1$ where V_0 (resp., V_1) is spanned by those weight vectors with weight $\omega - \gamma$ where γ has δ-height zero (resp., positive δ-height). We want to show that V_0 equals V_δ.

Similarly, write U_0 (resp., U_1) for the subgroup of G generated by $\mathfrak{X}_{-\rho}$, as ρ varies over the positive roots with δ-height zero (resp., positive δ-height). Equation (3.3) shows that V_0 is U_0-invariant, V_1 is U_1-invariant, and the induced action of U_1 on V/V_1 is trivial.

Let U denote the subgroup of G generated by the $\mathfrak{X}_{-\rho}$ as ρ varies over all the positive roots. Take a $u \in U(k)$ and write $u = u_1 u_0$ where u_i belongs to $U_i(k)$; this is possible by [Bor, 21.9]. We have

$$uv^+ = u_1 u_0 v^+ = u_0 v^+ + v_1 \quad \text{for some } v_1 \in V_1.$$

Because UB is dense in G [Bor, 14.14], every linear function on the G-orbit of v^+ also vanishes on the UB-orbit of v^+. It follows that the subspace UBv^+ is all of V. Since the line kv^+ is B-invariant, we have observed the standard fact that Uv^+ is all of V. In particular, the uv^+ from the preceding paragraph span V as u ranges over $U(k)$, and we have proved that V_0 equals $U_0 v^+$.

The same argument—with G, U, B replaced with L_δ, U_0, and the subgroup of G generated by $T \cap L_\delta$ and the $\mathfrak{X}_\rho$ for ρ a positive root of δ-height zero—shows that V_δ equals $U_0 v^+$. Hence V_δ equals V_0. □

We now address Question (2.1).

3.4 Proposition. *For every $\delta \in \Delta$, the subspace V_δ is a nonzero, proper subspace of V stabilized by P_δ.*

Proof. V_δ is clearly nonzero because it contains v^+. We now show that it is proper. The highest weight $\widetilde{\alpha}$ of Φ equals $\sum_{\alpha \in \Delta} c_\alpha \alpha$ with every c_α a natural number [Hu 80, 10.4A]. In particular, $\langle \omega, \widetilde{\alpha} \rangle = c_\beta$ is a positive integer. Since the set of weights of V is saturated [Bou Lie, §VIII.7.2], V contains a nonzero vector of weight $\omega - \widetilde{\alpha}$. But $\widetilde{\alpha}$ has positive δ-height, so $\omega - \widetilde{\alpha}$ is not a weight of V_δ.

Finally we show that V_δ is stabilized by P_δ. By [Bor, 14.18], P_δ is generated by four types of subgroups:

(1) the torus T. It normalizes L_δ and the line kv^+, hence also V_δ.

(2) groups $\mathfrak{X}_\rho$, for ρ a root with δ-height zero. These groups belong to L_δ and so stabilize V_δ.

(3) groups $\mathfrak{X}_\rho$, for ρ a positive root with nonzero (hence positive) δ-height. For α a sum of positive roots (possibly zero) with δ-height zero, the sum $\alpha - i\rho$ has negative δ-height for all $i \geq 1$, hence there are no nonzero vectors in V with weight $\omega - (\alpha - i\rho)$. Equation (3.3) gives that $\mathfrak{X}_\rho$ fixes V_δ elementwise.

(4) groups $\mathfrak{X}_{-\rho}$ for ρ a root with positive δ-height and of the form $\sum_{\gamma \in \Delta\setminus\{\delta\}} c_\gamma \gamma$. Since ρ is a root, the subset of Δ on which the coefficients c are nonzero is connected [Bou Lie, §VI.1.6], and since ρ's δ-height is positive, we find that ρ is a sum of simple roots γ that are not connected to the δ-component. Hence $\mathfrak{X}_{-\rho}$ commutes with L_δ. Finally, c_β is zero, so $\langle \omega, -\rho \rangle = 0$ and $\mathfrak{X}_{-\rho}$ kills v^+, that is, $\mathfrak{X}_{-\rho}$ kills V_δ.

□

Before treating Question (2.2) regarding incidence, we observe that we know a lot about the spaces V_δ. The case $\delta = \beta$ is trivial, so for the rest of this section we fix a $\delta \in \Delta$ that is not β.

3.5 Proposition. *For $\delta \in \Delta\setminus\{\beta\}$, the space V_δ is a fundamental irreducible representation of L_δ.*

Proof. Suppose that $v \in V_\delta$ is fixed by $\mathfrak{X}_\rho$ for every positive root ρ with zero δ-height. The argument in item (3) in the proof of Prop. 3.4 shows that v is fixed by $\mathfrak{X}_\rho$ for every positive root ρ. Since V is an irreducible representation of G, v is in the k-span of v^+.

The previous paragraph shows that kv^+ is the only highest weight line for V_δ relative to the maximal torus $T_\delta = T \cap L_\delta$ of L_δ. Since V_δ is a completely reducible representation of L_δ, it is irreducible.

The highest weight of V_δ is the restriction of ω to T_δ; we denote it by $\overline{\omega}$. Since ω is a fundamental weight of G, the restriction $\overline{\omega}$ is a fundamental weight of L_δ. □

The dimension of V_δ can be looked up in, e.g., [Bou Lie, chap. 8, Table 2].

3.6. Recall that the weights are partially ordered by setting $\lambda_1 \geq \lambda_2$ if $\lambda_1 - \lambda_2$ is a sum of positive weights. For example, we have already used that ω is the largest weight of V. On the other hand, $w_0\omega$ is the smallest weight, where w_0 is the longest element of the Weyl group, i.e., the one that sends Δ to $-\Delta$. (To see this, note that w_0 permutes the weights of V and reverses the partial ordering.)

We may apply the same observations to the irreducible representation V_δ of L_δ. Computing relative to the torus T_δ, the weights of V_δ lie between the highest weight $\overline{\omega}$ and the lowest weight $w_0\overline{\omega}$, where w_0 is the longest element in the Weyl group of L_δ. From the tables in [Bou Lie], one can quickly find the nonnegative integers k_α such that

$$w_0\overline{\omega} = \overline{\omega} - \sum k_\alpha \alpha$$

where α runs over the roots in the δ-component. Considering V_δ as a subspace of the representation V of G, Lemma 3.2 gives that the weights of V_δ are precisely those weights μ of V such that

$$\omega - \sum k_\alpha \alpha \leq \mu \leq \omega.$$

We close this section by answering the question of incidence—i.e., Question (2.2)—in the most important case. We say that a vertex in a graph is *terminal* if it is joined to at most one other vertex.

We call a subspace $X \in \Gamma_V$ of type δ—i.e., a subspace X in the $G(k)$-orbit of V_δ—a *δ-space*. (We allow here also the possibility that $\delta = \beta$.)

3.7 Proposition. *Let X be a δ-space in Γ_V (with $\delta \neq \beta$). Suppose that the δ-component of Δ is of type A and β is a terminal vertex of the δ-component.*

(1) Every 1-dimensional subspace of X is a β-space.

(2) If the δ-component contains the δ'-component, we have: X is incident to a δ'-space X' if and only if X contains X'.

Proof. To prove the proposition, we may conjugate X and so assume that X is actually V_δ. The group L_δ is a special linear group because the δ-component is of type A. Examining the highest weight of the representation V_δ of L_δ as in the proof of Prop. 3.5, we find that there is an isomorphism $L_\delta \xrightarrow{\sim} SL(V_\delta)$ that identifies the natural representations of L_δ and $SL(V_\delta)$ on V_δ. Since every 1-dimensional subspace of V_δ is in the $SL(V_\delta)$-orbit of V_β, this proves (1).

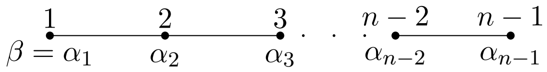
Now we prove (2). First suppose that X and X' are incident. That is, there is some $g \in G(k)$ such that conjugating the stabilizers of X and X' by g gives P_δ and $P_{\delta'}$ respectively. By the bijection between parabolics and objects in Γ_V, we find $gX = V_\delta$ and $gX' = V_{\delta'}$. Since $L_{\delta'}$ is contained in L_δ, clearly $V_{\delta'}$ is contained in V_δ, hence X' is contained in X.

Conversely, suppose that X' is contained in X. Applying an element of $G(k)$, we may assume that X equals V_δ. Since L_δ is $SL(V_\delta)$, all subspaces of V_δ with the same dimension are L_δ-equivalent. Hence X' is L_δ-equivalent to $V_{\delta'}$. The parabolics P_δ, $P_{\delta'}$ corresponding to V_δ, $V_{\delta'}$ contain the standard Borel B, hence are incident. ☐

4 Example: type A (projective geometry)

In this section, we describe the objects in the geometry Γ_V constructed from $G := SL_n$ as in §3 when the representation is the standard one on $V := k^n$, corresponding to the simple root $\beta := \alpha_1$. We "discover" that Γ_V is projective $(n-1)$-space. We could do this explicitly in terms of matrices as in Example 2.3, but such an argument would be hard to generalize to other groups. Instead, we give an algebraic-group- and representation-theoretic argument.

We defined V_β, a.k.a. V_{α_1}, to be the 1-dimensional subspace of V spanned by the highest weight vector. For $\alpha_i \in \Delta$ with $i \neq 1$, V_{α_i} is the standard representation of L_{α_i}. Therefore, the dimension of V_{α_i} is precisely i. We summarize this in the Dynkin diagram, where each vertex is labeled with α_i and $\dim V_{\alpha_i}$:



Since SL_n acts transitively on the i-dimensional subspaces of V for all i, we have: the α_i-spaces are the i-dimensional subspaces of V. By Prop. 3.7.2, two subspaces are incident if and only if one contains the other. This is the classical description of $(n-1)$-dimensional projective space as consisting of lines through the origin in k^n.

5 Strategy

In the next few sections, we will fix a split simply connected group G and give an explicit description of the geometry Γ_V. One imagines that the geometry Γ_V we have just constructed will be easiest to visualize if the ambient vector space V is small. With that in mind, we will focus on the case where V is the smallest irreducible representation of G. For G of type A, D_4, or E_6, there are multiple equivalent choices, and we arbitrarily pick one.

type of G	A_n	B_n	C_n	D_n	E_6	E_7	F_4	G_2
β	α_1	α_1	α_1	α_1	α_1	α_7	α_4	α_1
$\dim V$	$n+1$	$2n+1$	$2n$	$2n$	27	56	26	7

(5.1)

We number the elements of Δ as in the tables in [Bou Lie]. We remark that in all cases the root β is a terminal vertex of Δ as in the hypotheses for Prop. 3.7. We call a representation V as in the table above a *standard representation* of G. We have omitted type E_8, see 9.3 for comments. (We remind the reader that despite our focus on the standard representation of G, the recipe in §3 gives a concrete realization of Γ_P for every fundamental representation, and one can compute the dimensions of the δ-spaces using Prop. 3.5.)

Roughly speaking, each example consists of three parts: dimensions and properties, transitivity, and incidence.

In "dimensions and properties", for each $\delta \in \Delta$ we compute the dimension d of V_δ and some algebraic properties $\mathcal{P}$ satisfied by V_δ. These properties will be obviously G-invariant, hence they will be satisfied by all the δ-spaces. Here we restrict ourselves to the tools of elementary representation theory. This has two advantages. First, no special background is required to understand the exceptional groups versus the more-familiar classical groups. Second, we hope the reader will view our descriptions of the δ-spaces as reasonably canonical and not ad hoc.

In "transitivity", we prove:

> The group $G(k)$ acts transitively on the set of d-dimensional subspaces of V satisfying the properties $\mathcal{P}$. (5.2)

Since the δ-spaces are one $G(k)$-orbit by definition, this proves:

> The δ-spaces are precisely the d-dimensional subspaces of V satisfying the properties $\mathcal{P}$. (5.3)

In many cases, we will refer to the literature for a proof of (5.2). The proofs in the literature use various interpretations of the standard representation as the vector space underlying some algebraic structure. For example, in the type A example in §4, we used the fact that SL_n acts transitively on the subspaces of k^n of a given dimension.

In fact, it suffices to prove (5.2) in the case where k is algebraically closed together with the statement

> If X satisfies $\mathcal{P}$, then $X \otimes \overline{k}$ satisfies $\mathcal{P}$, (5.4)

where X is a d-dimensional subspace of V and $\overline{k}$ is an algebraic closure of k. To see this, suppose that X satisfies $\mathcal{P}$. By our two hypotheses, there is some $g \in G(\overline{k})$ such that $g(V_\delta \otimes \overline{k}) = X \otimes \overline{k}$. For every σ in the Galois group of $\overline{k}/k$, we find that

$$\sigma(g)(V_\delta \otimes \overline{k}) = \sigma(X \otimes \overline{k}) = X \otimes \overline{k} = g(V_\delta \otimes \overline{k}),$$

hence the stabilizers $\sigma(g)P_\delta\sigma(g)^{-1}$ and $gP_\delta g^{-1}$ agree. That is, $gP_\delta g^{-1}$ is invariant under every element of in the Galois group, so it is defined over k [Bor, AG.14.4] and there is some $h \in G(k)$ such that $hP_\delta h^{-1}$ equals $gP_\delta g^{-1}$ [Bor, 21.12]. This implies that $h(V_\delta \otimes \overline{k}) = g(V_\delta \otimes \overline{k}) = X \otimes \overline{k}$, hence hV_δ equals X.

In the examples below, $\mathcal{P}$ will almost always be a statement such as "a certain polynomial vanishes", for which property (5.4) clearly holds. In such cases, we will omit any discussion of (5.4). The only exception will be for certain spaces related to the E_6 geometry, see §7.

In "incidence", we give a concrete description of how to tell if a δ- and a δ'-subspace are incident. Our final description is purely in terms of subspaces of V, with no mention of the corresponding parabolic subgroups. In most cases, a δ- and δ'-subspace will be incident if and only if one contains the other. When this occurs, we will say that *incidence is the same as inclusion.* In this "incidence" portion, we return to the techniques of representation theory and eschew algebraic interpretations of the representation. We do not need to consider the case where δ equals δ', because two spaces of the same type are incident if and only if they are equal.

5.5 Remark. The hypothesis that k has characteristic zero smoothes the presentation, but is in some cases unnecessary. (The rest of this remark will not be used elsewhere in the paper, so we omit many details.) The material in Sections 1 and 2 made no use of the characteristic zero hypothesis. Regarding Section 3, it remains true in all characteristics that there is a unique irreducible module V with highest weight ω for each dominant weight ω, see e.g. [Bor, 24.4]. However, sometimes the weights and dimension of V are not what one would expect coming from characteristic zero. (For example, in characteristic 2 the standard representation of B_n has dimension $2n$, not $2n+1$.) On the positive side, Lemma 3.2 remains unchanged.

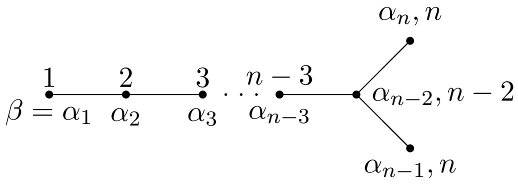
Moreover, all the results of §3 hold *regardless of the characteristic of* k when V is the standard representation of A_n, C_n, D_n, E_6, or E_7. In those cases, the highest weight ω is minuscule, i.e., all of the weights smaller than ω are Weyl-conjugates of ω, hence the weights of V are precisely the Weyl-orbit of ω, regardless of the characteristic. Moreover, the restricted weight $\overline{\omega}$ is a minuscule weight for L_δ for all $\delta \in \Delta \setminus \{\beta\}$.

6 Example: type D (orthogonal geometry)

Consider the split simply connected group G of type D_n with $n \geq 4$. This group is sometimes denoted Spin_{2n}. The geometry in this case is more complicated than for type A, apparently because the Dynkin diagram has a fork in it. This case will illustrate the basic principles involved in handling forking diagrams, and we will use them when treating the E-groups later. This geometry will be further investigated in Sections 11 and 12 below.

Dimensions and properties

As in the type A case, we compute the dimension of the V_δ's using Prop. 3.5.



6.1. The standard representation of G has dimension $2n$. Since $-w_0\omega_1$ is ω_1, where w_0 is the longest element of the Weyl group, there is a nondegenerate G-invariant bilinear form b on V, unique up to multiplication by an element of $k^\times$ [Bou Lie, 8.7.5, Prop. 12]. Moreover, b is symmetric [Bou Lie, chap. 8, Table 1]. For v^+ the highest weight vector in V and t an element of the maximal torus T, we have:

$$b(v^+, v^+) = b(t \cdot v^+, t \cdot v^+) = b(\omega(t)v^+, \omega(t)v^+) = \omega(t)^2 b(v^+, v^+).$$

Since ω is not the trivial character, $b(v^+, v^+)$ is 0. Traditionally, a subspace X is called *isotropic* if $b(X, X)$ is zero. We have just observed that the α_1-spaces are isotropic. By Prop. 3.7.1, the α_i-spaces are isotropic for every i.

6.2. **Connection with the special orthogonal group.** In order to prove transitivity, we will now relate G to the special orthogonal group $SO(b)$, the subgroup of $SL(V)$ preserving the bilinear form b. Since b is G-invariant and G is connected, the representation of G on V is a homomorphism $\rho\colon G \to SO(b)$; we claim that ρ is a central isogeny. Indeed, every proper, closed normal subgroup of G is central, hence $\ker \rho$ is finite and the image of ρ has the same dimension as G. Root system data gives that the dimension of the Lie algebra of G (equivalently, the dimension of G) is $\binom{2n}{2}$. On the other hand, the Lie algebra of $SO(b)$ is isomorphic over an algebraic closure of k to the space of skew-symmetric $2n$-by-$2n$ matrices, which also has dimension $\binom{2n}{2}$. Since $\operatorname{im} \rho$ and $SO(b)$ are connected and have the same dimension, they are the same. That is, ρ is surjective. The claim now follows because we are in characteristic zero, hence ρ is automatically separable.

Transitivity

We claim that G acts transitively on the m-dimensional isotropic subspaces of V for $m < n$. (We refer the reader to [Lam, Ch. I] for basic terminology and facts regarding symmetric bilinear forms.) Let X, X' be isotropic of dimension m. They each lie in a direct sum of m hyperbolic planes in V, and there is an isometry f of b that sends X to X' by Witt's Extension Theorem. Since V is isomorphic to a direct sum of n hyperbolic planes, there is at least one plane where we may choose f as we please. If f has determinant -1, we modify f by a hyperplane reflection in this "extra" hyperbolic plane so that f has determinant 1. Over an algebraic closure $\overline{k}$ of k, f is in the image of the map $G(\overline{k}) \to SO(b)(\overline{k})$, which proves the claim. Moreover, we have proved that the α_i-spaces are the i-dimensional isotropic subspaces for $1 \leq i \leq n-2$.

We claim that the n-dimensional isotropic subspaces of V make up two $G(k)$-orbits. Since the stabilizers of V_{α_n} and $V_{\alpha_{n-1}}$ are the non-conjugate subgroups P_{α_n} and $P_{\alpha_{n-1}}$, there are at least two G-orbits. On the other hand, consider the orthogonal group $O(b)$, i.e., the subgroup of $GL(V)$ preserving b; it is generated by $SO(b)$ and a hyperplane reflection. In particular, $SO(b)$ is a normal subgroup of index 2. An argument as in the preceding paragraph shows that $O(b)$ acts transitively on the set of n-dimensional isotropic subspaces, hence $SO(b)$ has at most two orbits. We have shown that

$$\{\alpha_n\text{- and } \alpha_{n-1}\text{-spaces}\} = \{\text{isotropic subspaces of dimension } n\}.$$

Incidence

6.3. Consider an α_i-space X_i and an α_j-space X_j with $i < j$. If (i, j) is not $(n-1, n)$, then Prop. 3.7.2 applies and incidence is the same as inclusion. We now argue that an α_{n-1}-space and an α_n-space are incident if and only if their intersection has dimension $n-1$.

Let M be the subgroup of G generated by the $\mathfrak{X}_\rho$ as ρ ranges over the roots with α_{n-1}- and α_n-height zero; it has type A_{n-2}. Write Y for the subspace Mv^+ of V; it is the standard representation of M by the reasoning in §3 and so has dimension $n-1$. It is contained in V_{α_n}, so it is isotropic.

The parabolic $P_{\{\alpha_{n-1},\alpha_n\}}$ stabilizes Y by the argument in the proof of Prop. 3.4. The only subgroups of G properly containing $P_{\{\alpha_{n-1},\alpha_n\}}$ are $P_{\alpha_{n-1}}$, P_{α_n}, and G, none of which stabilize Y. (For example, $P_{\alpha_n}Y$ contains $L_{\alpha_n}v^+$, which has dimension n.) Hence the stabilizer of Y is precisely $P_{\{\alpha_{n-1},\alpha_n\}}$.

Suppose first that X_{n-1} and X_n are incident, so there is a $g \in G(k)$ such that $gX_{n-1} = V_{\alpha_{n-1}}$ and $gX_n = V_{\alpha_n}$. The intersection $gX_{n-1} \cap gX_n$ contains Y and so has dimension at least $n-1$. Because X_{n-1} has dimension n and is not contained in X_n, this shows that the intersection has dimension exactly $n-1$.

Conversely, suppose that $X_{n-1} \cap X_n$ has dimension $n-1$. For $i = n-1, n$, there is a $g_i \in G(k)$ such that $gX_i = V_{\alpha_i}$ and $g_i(X_{n-1} \cap X_n) = Y$ by the argument in the proof of Prop. 3.7(2). Therefore the stabilizer $g_i P_{\alpha_i} g_i^{-1}$ of X_i contains the stabilizer of $X_{n-1} \cap X_n$, which is a parabolic subgroup of G. That is, X_{n-1} and X_n are incident.

6.4 Remark. A consequence of the above argument is that $V_{\alpha_{n-1}}$ and V_{α_n} are the unique α_{n-1} and α_n spaces containing Y. Indeed, if X is an α_i-space containing Y for $i = n-1$ or n, then the stabilizer of X contains $P_{\{\alpha_{n-1},\alpha_n\}}$ by the preceding paragraph, hence the stabilizer of X is P_{α_i}. The bijection between objects in Γ_V and their stabilizers gives that X equals V_{α_i}.

Since $G(k)$ acts transitively on the $(n-1)$-dimensional isotropic subspaces, we have proved: *Each $(n-1)$-dimensional isotropic subspace is the intersection of two uniquely determined and incident n-dimensional subspaces, one of type α_n and one of type α_{n-1}.* This is a standard result in quadratic form theory, usually proved by quadratic-form-theoretic methods, see e.g. [Ch, III.1.11].

6.5. **An alternative view.** We now outline the geometry that one obtains from G by considering the fundamental representation with highest weight α_n (a "half-spin" representation) instead of the standard representation. We continue with the same definitions of V, V_{α_i}, etc., as in the rest of this section. We may identify the vector space underlying the half-spin representation S with $\wedge^{\text{even}} V_{\alpha_n}$ as described in [Ch, chap. 3].

For $i \neq n-1$, the subspace V_{α_n} contains V_{α_i} and the ideal of $\wedge V_{\alpha_n}$ generated by $\wedge^i V_{\alpha_i}$ is stabilized by P_{α_i}. (Recall that the action of G on S is not precisely the standard action of G on $\wedge V$, see [Ch, §2.2]. In particular, G stabilizes $\wedge V_{\alpha_n}$.) Following the naive algorithm in §2, we take the intersection of $(\wedge^i V_{\alpha_i}) \wedge (\wedge V_{\alpha_n})$ with S to be the subspace corresponding to P_{α_i}. Thinking in terms of exterior powers of vector spaces, it is clear that the α_i-spaces in the half-spin representation have dimension 2^{n-i-1} for $i \leq n-2$;

they correspond to the right ideals in the even Clifford algebra constructed in [Ga 99, §1]. The α_n-spaces are 1-dimensional and are the "pure spinors" corresponding to even maximal isotropic subspaces in the language of [Ch, §3.1]. We do not know how to describe the α_{n-1}-spaces in this geometry.

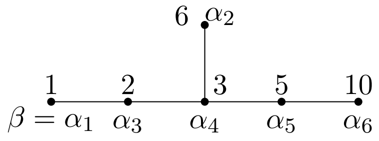
7 Example: type E_6

The geometry for the split simply connected group G of type E_6 exhibits two complexities. We are prepared for the first—the fork in the diagram—thanks to our work in the previous section on groups of type D. The second complication is new: the root $\delta = \alpha_6$ does not satisfy the hypotheses of our workhorse Prop. 3.7.

This is the last example we will do in detail. In the final section of this paper, we will give an explicit description of the duality in this geometry.

Dimensions and properties

The Dynkin diagram for E_6, labeled with the dimensions of the corresponding spaces is:



7.1. We will make a detailed study of the weights of representation V in order to understand the algebraic properties satisfied by the objects in the geometry. Recall from, e.g., [Bou Lie, §VIII.7.3] that a weight λ is *minuscule* if $\langle w\lambda, \alpha\rangle = 1$, 0, or -1 for every element w of the Weyl group and every root α. Further, ω_1 is a minuscule weight for E_6, and this implies that the weights of V are those weights in the Weyl orbt of ω_1 and all have multiplicity one.

Figure 7.2 displays the 27 weights of V as a Hasse diagram relative to the usual partial ordering of the weights (recalled in 3.6). An edge joining two weights $\lambda > \mu$ is labeled with i if $\lambda - \alpha_i = \mu$. The row vectors list the coordinates of the weights with respect to the basis consisting of fundamental weights. The lowest weight of V_{α_i} (cf. 3.6) is labeled λ_i. (This diagram is an elaboration of [PSV, Fig. 20]. To construct such a diagram from scratch, one calculates the Weyl orbit of ω_1—this amounts to a single instruction in the software packages LiE [vLCL] or Magma [BCP]. Alternatively, Dynkin [Dy,

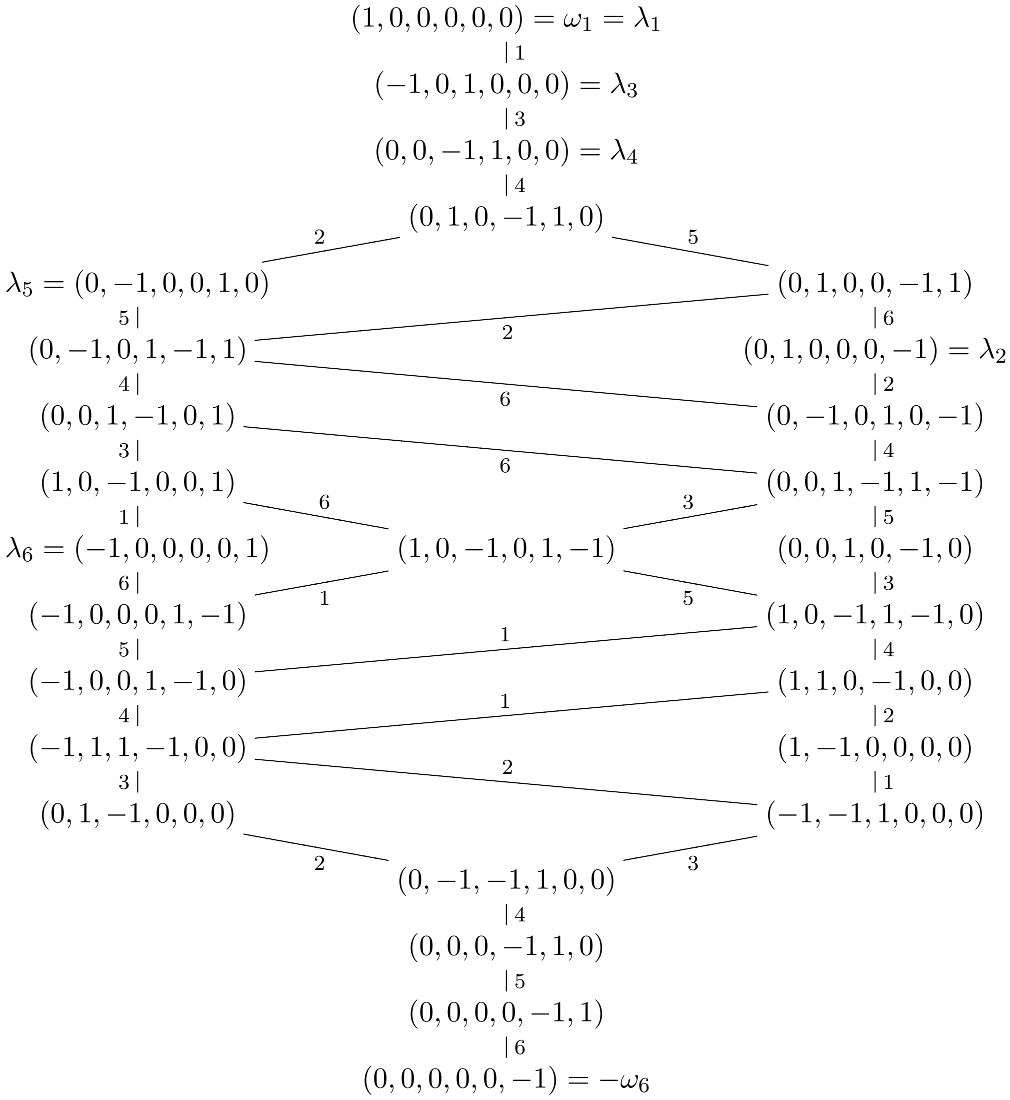


Figure 7.2: Hasse diagram of the weights of V, where $(c_1, c_2, \ldots, c_6)$ denotes the weight $\sum_{i=1}^{6} c_i\omega_i$. The map $-\phi$ reflects the diagram across its horizontal axis of symmetry.

p. 333] gives a general algorithm for finding the weights of an irreducible representation from its highest weight using [Bou Lie, VIII.7.2, Prop. 3(i)]. Next it is helpful to observe that all the weights are of the form $\omega_1 - \alpha$ for a sum of positive roots α, so the weights are stratified by the height of α.)

We remark that one can see from Figure 7.2 that ω_1 is minuscule; indeed, every coordinate of every weight is 1, 0, or -1, as required. We will make repeated use of this fact in §13.

The automorphism of order 2 of the Dynkin diagram gives an automorphism of the root system, hence an automorphism ϕ of G of order 2.

7.3 Lemma. *The subgroup of G consisting of elements fixed by ϕ is split simple of type F_4.*

The lemma is well-known. We sketch a proof for the reader's convenience.

Sketch of proof. Steinberg [St mem] gives that the fixed subgroup G_ϕ is connected (his 9.7) and reductive (his 9.4). We now inspect its Lie algebra $\mathfrak{g}_\phi$, which consists of the elements of the Lie algebra $\mathfrak{g}$ of G that are fixed by ϕ. Fix a Chevalley basis $\{x_\alpha \mid \alpha \in \Phi\} \cup \{h_\delta \mid \delta \in \Delta\}$ for $\mathfrak{g}$ as in [Hu 80, §8]. The elements $x_\alpha + \phi(x_\alpha)$ for $\alpha \in \Phi$ and $h_\delta + \phi(h_\delta)$ for $\delta \in \Delta$ span $\mathfrak{g}_\phi$ since ϕ has order 2. Moreover, the h_δ's are uniquely determined, hence $\phi(h_\alpha)$ equals $h_{\phi(\alpha)}$ for all $\alpha \in \Phi$. Computing the root space decomposition of $\mathfrak{g}_\phi$ with respect to the torus spanned by

$$h_{\alpha_1} + h_{\alpha_6},\ h_{\alpha_3} + h_{\alpha_5},\ h_{\alpha_4}, \text{ and } h_{\alpha_2}, \tag{7.4}$$

we find that $\mathfrak{g}_\phi$ is simple of type F_4, where the displayed elements correspond to the coroots $\check{\alpha}_4$, $\check{\alpha}_3$, $\check{\alpha}_2$, and $\check{\alpha}_1$ respectively. □

For simplicity, we denote the fixed subgroup by F_4. We can compute the restriction of weights of V to F_4 on the level of Lie algebras: A weight $\lambda := \sum c_i\omega_i$ maps $h_{\alpha_j} \mapsto c_j$, hence it maps the elements listed in (7.4) to

$$(c_1 + c_6),\ (c_3 + c_5),\ c_4, \text{ and } c_2$$

respectively. In terms of the fundamental weights of F_4, the restriction of λ is $c_2\omega_1 + c_4\omega_2 + (c_3 + c_5)\omega_3 + (c_1 + c_6)\omega_4$. We find that V decomposes (as a representation of F_4) as a direct sum of a 1-dimensional trivial representation C and the standard representation of F_4, which we denote by V_0.

7.5 Proposition. *There is a bilinear form b on V such that*

$$b(\phi(g)x, gy) = b(x, y) \quad \text{for all } g \in G \text{ and } x, y \in V. \tag{7.6}$$

It is unique up to multiplication by a scalar. Moreover, it is symmetric and nondegenerate, and $b|_C$ is not zero.

Proof. First we construct a bilinear form b on V satisfying (7.6). Write $\rho\colon G \to GL(V)$ for the representation of G on V, and write $\rho^*\colon G \to GL(V^*)$ for the dual representation defined by

$$(\rho^*(g)f)(x) := f(\rho(g)^{-1}x) \quad \text{for } g \in G,\ f \in V^*, \text{ and } x \in V.$$

The representations $\rho\phi$ and ρ^* are both irreducible with highest weight ω_6, hence they are isomorphic. Fix an isomorphism $h : V \to V^*$ such that $h\rho\phi(g)h^{-1} = \rho^*(g)$ for all $g \in G$. Define b by setting

$$b(x, y) := h(x)(y).$$

This b is clearly bilinear and

$$b(\phi(g)x, gy) = h(\rho\phi(g)x)(gy) = [\rho^*(g)h(x)]\,(gy) = b(x, y).$$

We now argue that any bilinear form b satisfying (7.6) is symmetric. Set

$$b_\varepsilon(x, y) := b(x, y) + \varepsilon b(y, x).$$

Then b_1 and b_{-1} are bilinear, b_1 is symmetric, b_{-1} is skew-symmetric, and $2b = b_1 + b_{-1}$. We prove that b_{-1} is identically zero. In any case, b_{-1} satisfies (7.6) (using that ϕ is its own inverse), hence b_{-1} is F_4-invariant. But V_0 does not support a nonzero F_4-invariant skew-symmetric form, hence b_{-1} restricts to zero on V_0. Fix $x \in V_0$ a nonzero vector with a nonzero weight λ with respect to F_4, and let c be a nonzero vector in C. Since

$$b_\varepsilon(c, x) = b_\varepsilon(tc, tx) = \lambda(t) b_\varepsilon(c, x)$$

for t in the F_4-torus, $b_\varepsilon(c, x)$ is zero. Since V_0 is an irreducible representation of F_4, $b_\varepsilon(c, V_0)$ is zero. But b_{-1} is skew-symmetric, so $b_{-1}(c, c)$ is also zero, and we have proved the claim.

The previous paragraph also gives more. Continue the assumption that b satisfies (7.6) and suppose that b is not identically zero. Then C and V_0 are orthogonal subspaces. For $x \in V$ and r in the radical of b, we have

$$b(gr, x) = b(r, \phi(g)^{-1}x) = 0,$$

hence the radical is G-invariant. Since V is irreducible and b is not identically zero, the radical is zero, i.e., b is nondegenerate. Since C is 1-dimensional and orthogonal to V_0, we find that b restricts to be nonzero on C.

We now prove uniqueness. Let b, b' be bilinear forms on V satisfying (7.6). The representation V_0 of F_4 supports a unique symmetric bilinear form up to a scalar multiple, so by modifying b' by a factor in $k^\times$, we may assume that b and b' have the same restriction to V_0. Then $b-b'$ is a bilinear form on V satisfying (7.6) that restricts to be zero on V_0. By the previous paragraph, $b - b'$ is identically zero, and we have proved uniqueness. $\square$

7.7. We can use representation theory to find G-invariant polynomial functions on V. Plugging the formal character of the dual representation V^* into the degree d complete symmetric polynomial in $\dim V^*$ variables gives the formal character of $S^d(V^*)$, the d-th symmetric power of V^*. From this, one can write $S^d(V^*)$ as a direct sum of irreducible representations [Hu 80, 22.5A]. (For small d, these computations are easily done using a computer package like LiE or Magma.) If $S^d(V^*)$ has a unique 1-dimensional summand—corresponding to a summand with highest weight 0—then V supports a G-invariant homogeneous polynomial of degree d, uniquely determined up to a factor in $k^\times$. This happens for G of type E_6 and $d = 3$.

Write N for a nonzero cubic form on V as discovered in the previous paragraph. We abuse notation and write N also for the trilinearization of N on V such that $N(x, x, x) = 6N(x)$ for all $x \in V$. Let # denote the bilinear product defined implicitly by the formula

$$b(x\#y, z) = N(x, y, z) \quad \text{for } x, y, z \in V. \tag{7.8}$$

Using (7.6), we find

$$\phi(g)(x\#y) = (gx)\#(gy) \quad \text{for } g \in G \text{ and } x, y \in V. \tag{7.9}$$

We write $x^\#$ for $(x\#x)/2$. (The factor 6 above arises naturally from multilinearization, see [Bou Alg, §IV.5.8, Prop. 12(i)]. The factor 2 here occurs for the same reason: In order to apply identities from Jordan theory in §13, we adopt the Jordan theorist view that the square $x \mapsto x^\#$ is the base object and the bilinear product is obtained by multilinearizing it.)

7.10. The same argument as in 6.1 shows that the α_1-spaces are 1-dimensional subspaces consisting of elements $x \in V$ such that $x^\# = 0$. We say a nonzero vector $x \in V$ is *singular* if $x^\# = 0$. (These 1-dimensional subspaces are precisely the singular points for the hypersurface in $\mathbb{P}(V)$ defined by $N = 0$, because $b(x^\#, y)$ is the directional derivative of N at x in the direction y.) We call a subspace of V singular if its nonzero elements are singular. By Prop. 3.7.1, the α_i-spaces are singular for $i \neq 6$.

We will now investigate the restriction of the representation of G on V to the subgroup L_{α_6} of type D_5. This will give us finer information about the product # and lead us to a description of the α_6-spaces. To see how a weight of G restricts to L_{α_6}, one drops the last coordinate and moves the second coordinate to the end of the vector (to allow for the fact that weights of D_5 and E_6 are numbered somewhat incompatibly in [Bou Lie]).

Let W' be the subgroup of the Weyl group W of G generated by the reflections with respect to the roots α_i for $i \neq 6$. It is the Weyl group of L_{α_6}, and it is the stabilizer of $-\omega_6$ in W [Hu 90, Th. 1.12c].

7.11 Lemma. *The orbits of W' in the weights of V are the weights $\geq \lambda_6$, the weights between λ_2 and $(0,0,0,0,-1,1)$, and the weight $-\omega_6$.*

Proof. Since the highest weight ω_1 of V is minuscule, we have $\langle \mu, \alpha \rangle = 1$, 0 or -1 for every weight μ of V and every root α. If μ and $\mu - \delta$ are both weights for some $\delta \in \Delta$, then $\langle \mu, \delta \rangle = 1$, $\langle \mu - \delta, \delta \rangle = -1$, and the reflection s_δ with respect to the root δ interchanges μ and $\mu - \delta$. Consulting Figure 7.2, we see that W' acts transitively on each of the three sets of weights named in the statement of the lemma.

Conversely, ω_1 and λ_2 restrict to the weights $(1,0,0,0,0)$ and $(0,0,0,0,1)$ on L_{α_6}, which are not congruent modulo the D_5 root lattice. Therefore, they lie in different W'-orbits [Bou Lie, VI.1.9, Prop. 27]. □

By restricting the weights of V to L_{α_6}, we can decompose V as a direct sum of irreducible representations. The proof of Lemma 7.11 shows that the components of V are

- the standard representation V_{α_6} of L_{α_6} (with highest weight $(1,0,0,0,0)$),
- a half-spin representation (with highest weight $(0,0,0,0,1)$), and
- a 1-dimensional trivial representation (from the lowest weight vector $-\omega_6$).

7.12 Corollary. *The Weyl group of type E_6 acts transitively on triples μ_1, μ_2, μ_3 of weights of V such that $\mu_1 + \mu_2 + \mu_3 = 0$.*

Proof. The Weyl group acts transitively on the weights of V, so we may assume that μ_1 is $-\omega_6$. Since $\mu_3 = \omega_6 - \mu_2$ is a weight, μ_2 cannot have last coordinate equal to -1, otherwise μ_3 would have last coordinate -2, which is impossible. In particular, μ_2 cannot be $-\omega_6$ or λ_2. Since the set of triples $-\omega_6, \mu_2, \mu_3$ with sum 0 is stable under the action of W', μ_2 must lie in the W'-orbit with lowest weight λ_6. □

Our preliminary results about the action of the Weyl group can now give us concrete information about the product #.

7.13 Lemma. *Let* x_1, x_2 *be nonzero vectors in* V *of weight* μ_1, μ_2 *respectively. The product* $x_1 \# x_2$ *is nonzero if and only if* $\phi(\mu_1 + \mu_2)$ *is a weight of* V.

The equations

$$\phi(\lambda_2 + \lambda_5) = \lambda_3 \quad \text{and} \quad \phi(\omega_1 + \lambda_6) = \omega_1 \tag{7.14}$$

furnish specific examples where the product # is not zero.

Proof of Lemma 7.13. If $\phi(\mu_1 + \mu_2)$ is not a weight of V, then the product is zero by (7.9). So suppose that $\phi(\mu_1 + \mu_2)$ is a weight of V. Since $-\phi$ is in the Weyl group, $\mu_3 := -\mu_1 - \mu_2$ is a weight of V; let y be a nonzero vector with that weight.

We claim that $N(x_1, x_2, y)$ is not zero, and hence that $x_1 \# x_2$ is not zero. Recall that every weight of V has multiplicity 1 because V is minuscule. Fix a basis $\{b_\lambda\}$ for V where b_λ has weight λ, and write N in terms of the dual basis $\{f_\lambda\}$. Since N is G-invariant, a monomial $f_{\nu_1} f_{\nu_2} f_{\nu_3}$ has zero coefficient if $\nu_1 + \nu_2 + \nu_3$ is not zero. On the other hand, since N is not identically zero, there exist weights ν_1, ν_2, ν_3 such that the coefficient of $f_{\nu_1} f_{\nu_2} f_{\nu_3}$ is not zero. Since their sum $\nu_1 + \nu_2 + \nu_3$ is zero, there is an element w in the Weyl group such that $w\nu_i = \mu_i$ for each i by Cor. 7.12. A representative of w can be found in G, hence the coefficient of $f_{\mu_1} f_{\mu_2} f_{\mu_3}$ in N is not zero. In particular, $N(x_1, x_2, y)$ is not zero, as claimed. □

Finally, we can give an explicit description of the α_6-space V_{α_6}.

7.15 Proposition. $V_{\alpha_6} = v^+ \# V$, *where* v^+ *is the highest weight vector.*

Proof. ($\supseteq$): Suppose that $x \in V$ has weight μ, which is necessarily at least the minimum weight $-\omega_6$. Since ϕ respects the partial ordering on the weights, $v^+ \# x$ has weight at least $\phi(\omega_1 - \omega_6) = \lambda_6$. But this is the lowest weight of V_{α_6}, so V_{α_6} contains $v^+ \# x$ by 3.6. Since every element of V is a sum of weight vectors, we have shown that V_{α_6} contains $v^+ \# V$.

($\subseteq$): Let $v \in V_{\alpha_6}$ be nonzero with weight μ. By Lemma 7.11, μ equals $w\lambda_6$ for some $w \in W'$; fix an element $n \in L_{\alpha_6}$ representing w. For v^- a vector of the lowest weight $-\omega_6$, the vector $n(v^+ \# v^-)$ is nonzero (by Lemma 7.13) of weight μ. Since all weight spaces in V are one-dimensional, $n(v^+ \# v^-)$ is a nonzero multiple of v. This proves that V_{α_6} is contained in $v^+ \# V$. □

For x singular in V, we call the subspace $x\#V$ a *hyperline*, following the terminology from [T 57, p. 25] and [Sc, p. 606]. Combining the proposition with (7.9), we find that every α_6-space is a hyperline.

7.16 Remark (Cf. [Fa, p. 17]). As the standard representation of the group L_{α_6} of type D_5, the space V_{α_6} supports an L_{α_6}-invariant quadratic form, uniquely determined up to a scalar. We claim that it is the form q given implicitly by the equation

$$x^{\#} = q(x)v^{+} \quad \text{for } x \in V_{\alpha_6}. \tag{7.17}$$

First, observe that for y, z weight vectors in V_{α_6}, $y\#z$ has weight at least

$$\phi(2\lambda_6) = \omega_1 - (\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6).$$

But the only edge leaving ω_1 in Fig. 7.2 is α_1, so ω_1 is the only possible weight for $y\#z$. Therefore, for every $x \in V_{\alpha_6}$, the vector $x^{\#}$ is in the k-span of v^{+} and the recipe (7.17) defines a quadratic form on V_{α_6}.

For $g \in L_{\alpha_6}$, we have $q(gx)v^{+} = q(x)\phi(g)v^{+}$. Since $\phi(g)$ fixes v^{+} (by 3.2, if you like), the form q is L_{α_6}-invariant. Moreover, q is not the zero form since $\#$ is bilinear and $v^{+}\#x$ is not zero for x of weight λ_6, by Lemma 7.13. Combining the two previous sentences, q is the unique invariant quadratic form as claimed, and it is nondegenerate. One consequence of this is that $v^{+}\#V$ cannot contain an isotropic 6-dimensional subspace. In terms of the geometry, an α_6-space cannot contain an α_2-space. A second consequence is that $(v^{+}\#V)^{\#}$ is kv^{+}; it follows that every α_6-space X equals $X^{\#}\#V$.

7.18. **Connections with Jordan algebras.** It is well-known that the cubic form N can be realized as the generic norm ("determinant") on the split Albert algebra A, see e.g. [SV 00, 7.3.1]. (Recall that an Albert algebra is a 27-dimensional exceptional Jordan algebra, see e.g. [SV 00, Ch. 5].) That is, there is a vector space isomorphism $f\colon V \to A$ such that the generic norm N_A on A satisfies

$$N_A(f(x)) = N(x) \quad \text{for } x \in V. \tag{7.19}$$

We now show that we can pick f such that (7.19) holds and such that f identifies b with the trace bilinear form b_A on A. This turns out to be a bit technical, but we need it in order to apply results about Albert algebras in the "transitivity" portion. We assume that k is algebraically closed.

Write b' for the symmetric bilinear form on V induced by b_A via f. Formula (7.6) with b' instead of b defines an automorphism ϕ' of $GL(V)$. This ϕ' restricts to an automorphism of G of order 2 by [Jac 3, p. 76], and it leaves a Borel subgroup B' and a maximal torus T' invariant by [St mem, 7.5]. There is some $h \in G(k)$ such that $hTh^{-1} = T'$ and $hBh^{-1} = B'$. Replacing f with the map $x \mapsto f(hx)$ replaces ϕ' with the map

$$g \mapsto h^{-1}\phi'(hgh^{-1})h.$$

Changing f in this way, we may assume that ϕ and ϕ' both leave T and B invariant.

Equation (7.6) implies that ϕ and ϕ' have the same action on the center of G, which consists of cube roots of unity. Therefore, $\phi'(g) = t\phi(g)t^{-1}$ for some $t \in T(k)$ [Bor, 14.9]. Elementary computations as in [KMRT, 2.7] show that $\phi(t) = t^{-1}$. Since k is algebraically closed, there is a "square root" s of t, i.e., an element $s \in T(k)$ such that $s^2 = t$ and $\phi(s) = s^{-1}$. Replacing f with the map $x \mapsto f(sx)$, we may assume that ϕ' equals ϕ. Prop. 7.5 gives that b' is a scalar multiple of b. But b was only defined up to a scalar multiple to begin with, so we may replace b with b' and we have found an isomorphism f as desired.

Transitivity

It is known that G acts transitively on the i-dimensional singular subspaces for $i = 1, 2, 3, 4$, and 6 by [SV 68, 3.12], [Fa, p. 33], [Ra, p. 35, Cor. 5], or [A E6, 6.5(2)]. Thus the α_1-, α_2-, α_3-, and α_4-spaces are as described in Table 7.20 below. Every hyperline is by definition of the form $x \# V$ for a singular $x \in V$. Since G acts transitively on the 1-dimensional singular subspaces, it acts transitively on the hyperlines by (7.9). Therefore, the α_6-spaces are the hyperlines.

We now prove that the α_5-spaces are the 5-dimensional, maximal singular subspaces. In the notation of (5.2)–(5.4), we take $d = 5$ and $\mathcal{P}$ to be "maximal singular". We have to be a little careful; for example, it is not clear that property (5.4) holds, that is, that the property "maximal singular subspace" is preserved when one goes from k to $\overline{k}$.

As in 6.3, take M to be the subgroup of G generated by $\mathfrak{X}_\rho$ as ρ varies over the roots of α_2- and α_6-height zero; it is of type A_4. Put $Y := Mv^+$; it is the standard representation of M and has dimension 5. It is contained in V_{α_2}, so it is singular. The arguments in 6.3 show that the stabilizer of Y is $P_{\{\alpha_2,\alpha_6\}}$.

Suppose first that k is algebraically closed. By [Ra, p. 35, Cor. 5], the collection of 5-dimensional singular subspaces of V is a union of two G-orbits, corresponding to those singular subspaces that are and are not maximal. Since V_5 and Y have non-conjugate stabilizers P_{α_5} and $P_{\{\alpha_2,\alpha_6\}}$, they lie in different orbits. This proves (5.2) for k algebraically closed.

Now we treat the case where k is arbitrary by proving (5.4) for the α_5-spaces. That is, let X be a 5-dimensional maximal singular subspace of V. For sake of contradiction, suppose that $X \otimes \overline{k}$ is contained in a 6-dimensional singular subspace $\overline{Z}$ of $V \otimes \overline{k}$. We note that the arguments in 6.3 show that $V_{\alpha_2} \otimes \overline{k}$ is the unique α_2-space containing $Y \otimes \overline{k}$; since $G(\overline{k})$ acts transitively on the 5-dimensional nonmaximal singular subspaces, every such subspace (e.g., $X \otimes \overline{k}$) is contained in a unique 6-dimensional singular subspace. For every σ in the Galois group of $\overline{k}/k$, we have $X \otimes \overline{k} = \sigma(X \otimes \overline{k}) \subseteq \sigma(\overline{Z})$. The uniqueness of $\overline{Z}$ implies that $\overline{Z}$ is stable under the Galois group, hence $\overline{Z}$ equals $Z \otimes \overline{k}$ for some singular subspace Z of V [Bor, AG.14.2]. That is, X is not a maximal singular subspace of V. This proves (5.4) for X of type α_5, hence also the claimed description of the α_5-spaces.

We summarize the descriptions of the α_i-spaces in Table 7.20.

__-space	description	name in [SV 68]
α_1	1-dim'l singular	point
α_2	6-dim'l singular	max'l space of 2nd kind
α_3	2-dim'l singular	space of prdim 1
α_4	3-dim'l singular	space of prdim 2
α_5	5-dim'l, maximal singular	max'l space of 1st kind
α_6	hyperline	line

Table 7.20: α_i-spaces in the E_6 geometry

Incidence

7.21. Let X' be an α_i-space and let X be an α_j-space with $i < j$. If (i, j) is not $(2, 5)$ or $(2, 6)$, Prop. 3.7.2 applies and incidence is the same as inclusion. As in the D_n case, we quickly find: An α_2- and an α_5-space are incident if and only if their intersection is 4-dimensional (equivalently, the stabilizer of their intersection is a parabolic of type $\{\alpha_2, \alpha_5\}$). An α_2- and an α_6-space are incident if and only if their intersection is 5-dimensional (equivalently, the stabilizer of their intersection is a parabolic of type $\{\alpha_2, \alpha_6\}$).

Bibliographic remarks. We deduced the existence of a G-invariant cubic form on V purely by computations with characters, but the invariant cubic form was written down long before these mathematical tools were available. For example, it is in Cartan's 1894 thesis, see [Ca th, p. 143]. Dickson pointed out in [Di] that the variables occurring in the cubic form correspond to the the 27 lines on a cubic surface. For a modern explanation of the correspondence between the 27 lines on a cubic surface and the 27 weights in Figure 7.2, see [Ma, §§23, 25] or [IS, pp. 235, 236]. Neher [N, §II.5] describes the correspondence from a Jordan-theoretic viewpoint. Lurie derives an explicit formula for the cubic form in [Lu].

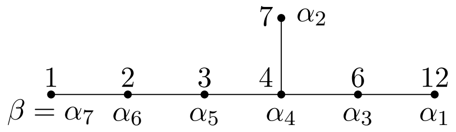
Diagrams like Figure 7.2 for different groups and representations can be found in, for example, [PSV], [Sc, §2.5], and the references listed on p. 519 of [Sc].

8 Example: type E_7

We will now treat the split group of type E_7. In the interest of brevity, this section is less detailed than the preceding ones.

Dimensions and properties

For G of type E_7, the Dynkin diagram looks like



(8.1)

As in 7.7, we find that there is a G-invariant quartic form q and a skew-symmetric bilinear form b on V; both are unique up to a factor in $k^\times$. We write q also for the quadralinear form obtained from q and define a symmetric trilinear map $t\colon V \times V \times V \to V$ (i.e., a linear map $S^3V \to V$) implicitly via

$$q(x, y, z, w) = b(x, t(y, z, w)) \quad \text{for } x, y, z, w \in V.$$

Since q has degree at least 3 and is preserved by the infinite group $G(k)$, one knows that the hypersurface in $\mathbb{P}(V)$ defined by the equation $q = 0$ is singular, see e.g. [OS, §6]. Consider now the singular locus S; it is a proper subvariety, consisting of the lines kx such that $t(x, x, x)$ is zero. We can ask if S is itself smooth. The rows of the Jacobian matrix at kx are of the form $t(x, x, y)$ as y varies over a basis of V. We say that x is *is rank one* if it is nonzero and the Jacobian matrix has rank ≤ 1, i.e., the image of the linear map $y \mapsto t(x, x, y)$ has dimension ≤ 1.

8.2 Example. The highest weight vector $v^+ \in V$ is rank one. Indeed, let $y \in V$ be a weight vector and write its weight as $\omega - \alpha$ where α is a sum of positive roots. If $t(v^+, v^+, y)$ is not zero it is a weight vector with weight $3\omega - \alpha$, necessarily equal to $\omega - \alpha'$ for some sum of positive roots α'. However, the lowest weight of V is $w_0\omega$ where w_0 is the longest element of the Weyl group of G, so α is at most $\omega - w_0\omega$. Putting these observations together with the fact that $w_0\omega = -\omega$, we have:

$$2\omega = \alpha - \alpha' \leq \alpha \leq 2\omega.$$

Hence $\alpha = 2\omega$ and $\alpha' = 0$. In particular, $t(v^+, v^+, V)$ is contained in the span of v^+.

Note the contrast with the situation for E_6. In that case, the group acts transitively on the singular locus, hence the singular locus is smooth.

8.3 Remark. Bringing to bear the explicit formula for $t(x, x, y)$ from [Br E7, p. 88] and representatives of the orbits for the group action from [Kr, Th. 29], one can calculate that the Jacobian of S at a point kx has rank 1 or 12.

We claim that the δ-spaces are inner ideals for all $\delta \in \Delta$. An *inner ideal* is a subspace X of V such that $t(X, V, X)$ is contained in X. It suffices to check that V_δ is an inner ideal. The lowest weight for the action of G on V_δ is $\omega_7 - \alpha$, where α is a sum of simple roots in the δ-component, and the lowest weight for V is $-\omega_7$. Therefore, every weight of $t(V_\delta, V_\delta, V)$ is at least $\omega_7 - 2\alpha$. But every weight of V that differs from ω_7 by a sum of simple roots in the δ-component belongs to V_δ by Lemma 3.2. We have proved that V_δ is an inner ideal.

By Prop. 3.7.1, the α_i-spaces consist of rank one elements except possibly for the α_1-spaces. We say that an inner ideal is rank one if it consists of rank one elements.

Transitivity

The group G acts transitively on the 1-dimensional subspaces of V spanned by rank one elements by [Fe, 6.2, 7.7], that is, the α_7-spaces are the 1-dimensional rank one inner ideals.

The roots of E_7 not involving α_7 form a root system of type E_6; we write E_6 for the corresponding subgroup of G. Restricting the weights of V to E_6, we find that V is—as an E_6-module—a direct sum of the standard representation of E_6, its dual, and two 1-dimensional trivial representations. By [Ga 01, 6.12], every d-dimensional rank one inner ideal is in the G-orbit of one that is a direct sum of one of the 1-dimensional representation and a singular inner ideal of dimension $d - 1$ in the standard representation of

E_6. By the transitivity results from §7, for $i = 2, 4, 5, 6$ the α_i-spaces are the rank one inner ideals of the dimensions specified in (8.1). Similarly, the α_3-spaces are the maximal rank one inner ideals of dimension 6.

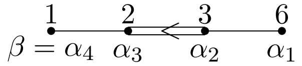
The group G acts transitively on the collection of 12-dimensional inner ideals by [Ga 01, 6.15], hence the α_1-spaces are the 12-dimensional inner ideals.

For the sake of brevity, we omit the “incidence” portion of this example.

9 Loose ends

To complete our discussion of the concrete realization of the geometries, we address some loose ends. Above, we have skipped the groups of type B and C; the reader should have no trouble filling them in from the previous examples.

9.1. Example: type F_4. We now sketch the case where G is of type F_4. We find the following diagram:



Klimyk’s provides a G-invariant bilinear product on V, which we denote by #. (We saw the objects G, V, # in §7, where they were known as F_4, V_0, #.) The usual argument shows that the product is identically zero on the α_4-spaces, and it is zero on the α_3- and α_2-spaces by Prop. 3.7.1.

The lowest weight of V_{α_1} is

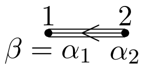
$$\omega_4 - (2\alpha_4 + 2\alpha_3 + \alpha_2) = \alpha_1 + \alpha_2 + \alpha_3$$

by 3.6. For x, y weight vectors in V_{α_1}, the product $x\#y$ has weight at least $2\alpha_1 + 2\alpha_2 + 2\alpha_3$. In particular, that character is not a weight of V since the α_1-coordinate of ω_4 is 1. Therefore, the product # is identically zero on the α_1-spaces also. Freudenthal calls α_1-spaces “symplecta”, since they are associated with the standard representation of a group of type C_3.

The group G acts transitively on the d-dimensional subspaces of V on which the product # is identically zero for $d = 1$ (the α_4-spaces) by [Fr, 28.27] or [A E6, 8.6], $d = 2, 3$ by [A E6, 9.5, 9.8], and for $d = 6$ (the α_1-spaces) by [Fr, 28.22]; alternatively, all four cases are treated by [Ra, Th. 2]. For all of the objects in Γ_V, incidence is the same as inclusion. Aside from α_1-spaces, this is Prop. 3.7.2. For cases involving an α_1-space, one can adapt the proof of 3.7.2, using the fact that a group of type C_3 acts transitively on d-dimensional isotropic subspaces of the standard representation for $d = 1, 2, 3$.

The space V may be interpreted as the trace zero elements in an Albert algebra, i.e., a 27-dimensional simple exceptional Jordan algebra; this identifies G with the group of automorphisms of the algebra. (The papers [Fr] and [Ra] cited above use this viewpoint.) An element x in such an Albert algebra has square zero if and only if the trace of x is zero (i.e., x is in V) and $x\#x = 0$, as can be easily seen using the "sharp expression" and "spur formula" from [McC book]. Therefore, the objects of Γ_V are precisely the subspaces in the Albert algebra of dimensions 1, 2, 3, and 6 on which the multiplication is identically zero.

9.2. Example: type G_2. The geometry for a group G of type G_2 is similar to that for type F_4, but everything is easier. The dimensions are summarized in the following diagram:



As in the F_4 case, there is a G-invariant bilinear product on V, which we denote by $\#$. The usual argument shows that the product is identically zero on the α_1-spaces, hence also on the α_2-spaces by Prop. 3.7.

The vector space V may be viewed as the trace zero elements in the split octonion algebra; this identifies G with the group of automorphisms of the algebra [SV 00]. As in the F_4 case, an element x in the split octonion algebra has square zero if and only if x is in V and $x\#x$ is zero. It is easy to prove that G acts transitively on the 1-dimensional subspaces of V on which the multiplication is zero using [SV 00, 1.7.3]. The Cayley-Dickson process gives an explicit description of the octonion algebra, which one can use to prove that G acts transitively on the 2-dimensional subspaces on which the multiplication is zero. (This essentially solves Problem 23.54 in [FH], cf. 9.4.) Aschbacher gives a different proof in [A G2].

In summary, the α_i-spaces are the i-dimensional subspaces of V on which the product $\#$ is zero. Incidence is the same as inclusion by Prop. 3.7.2.

9.3. Example: type E_8. The recipe in §3 for giving a concrete realization of the geometry associated with a group has been very effective with the examples considered so far. But what of the least familiar case, where G has type E_8? The recipe still works, of course, and for each fundamental representation V, it is easy to write down the dimension of the δ-spaces for each $\delta \in \Delta$. This is already interesting. But a problem occurs when we attempt to describe the algebraic properties that characterize the δ-spaces.

For example, the smallest fundamental representation V of G is the adjoint representation, with highest weight ω_8 and dimension 248. Representation theory does not provide any obvious additional structure on V, e.g., there is no G-invariant quintic form. Therefore, the only description of the δ-spaces that suggests itself is in terms of the Lie algebra structure.

The next smallest fundamental representation V has highest weight ω_1 and dimension 3875. This representation has G-invariant bilinear and cubic forms, each determined uniquely up to a nonzero scalar multiple. Thus there is also a G-invariant commutative product on V. Unfortunately, one cannot simply translate the analysis in §7 to this case. For example, the proof of Lemma 7.13 does not translate because the weights of V are not all one orbit under the Weyl group and some of the weights occur with multiplicity greater than one.

9.4. **Projective homogeneous varieties.** The above examples can all be viewed from the perspective of projective homogeneous varieties, i.e., projective varieties Y on which G acts transitively. We maintain our assumptions that G is split simply connected and V is a fundamental irreducible representation as in §3.

There is a bijection between subsets of Δ and isomorphism classes of projective homogeneous varieties given by sending $S \subseteq \Delta$ to $Y_S := G/P_S$. For example, $Y_\emptyset$ is a point because $P_\emptyset$ is all of G.

For S a singleton, say $\{\delta\}$, the k-points of Y_S are the δ-spaces in V. Indeed, the δ-spaces are defined to be the orbit of V_δ in the appropriate Grassmannian, and V_δ has stabilizer P_δ.

9.5 Example. For G of type B or D, the variety Y_{α_1} is a conic. The other Y_δ's are families of linear subspaces of the conic.

For an arbitrary subset $S \subseteq \Delta$, a *flag of type* S is a collection of pairwise incident subspaces $\{X_s \mid s \in S\}$ where X_s is an s-space. The flags of the extreme type Δ are called *chambers*. We call $\{V_s \mid s \in S\}$ the *standard flag of type* S. (What we call the standard chamber is traditionally called the "fundamental chamber".) We need the following consequence of the fact that Γ_V is a building:

9.6 Proposition. [T 74, 3.16] *Every flag of type S in Γ_V is contained in a chamber and is in the G-orbit of the standard flag of type S.*

In particular, G acts transitively on the collection of flags of type S. The stabilizer of the standard flag is the intersection $\cap_{s \in S} P_s$, which is P_S

[Bou Lie, IV.2.5, Th. 3c]. Hence the k-points of Y_S are the flags of type S in Γ_V.

We view the examples in the preceding sections as giving explicit descriptions of the geometry Γ_V as well as the projective homogeneous varieties under split groups G. When G is not split, the situation is somewhat more complicated. The absolute Galois group of k acts on the Dynkin diagram Δ, and there is a bijection between Galois-invariant subsets S of Δ and projective homogeneous varieties defined over k. The description in the split case can be altered to give a description in the general case. For groups of "inner type" A_n, one finds the generalized Severi-Brauer varieties as in [KMRT, §1]. (Note that these varieties may have no k-points.) For groups of type D_n, see [Ga 99] ($n = 4$) and [MPW, p. 183] ($n \neq 4$).

10 Outer automorphisms

Let Γ_V be a geometry defined from an irreducible representation V of G by the recipe in §2. Every automorphism ϕ of G permutes the parabolic subgroups, hence induces an automorphism of Tits's geometry Γ_P. Further, ϕ induces an automorphism of the concrete geometry Γ_V via the isomorphism between Γ_P and Γ_V from §2.

Every $g \in G(k)$ defines an automorphism of G by sending $h \mapsto ghg^{-1}$. Such an automorphism is called *inner*. It preserves types of objects and sends $X \in \Gamma_V$ to gX. In classical projective geometry, such an automorphism is called a *collineation*.

On the other hand, some groups have automorphisms that are not of this type; such automorphisms are called *outer*. They have a more interesting action on the geometry Γ_V in that they do not preserve the types of objects. In classical projective geometry, they are called *correlations*. As an example, the map $g \mapsto (g^{-1})^t$ is an automorphism of SL_3, and it is outer because it does not fix the center elementwise. We will see in Example 10.2 below that the induced map ψ is the polarity with respect to a certain conic.

Generally speaking, the existence of an outer automorphism of G implies a principle of duality (for D_4, triality) in the geometry Γ_V. For SL_3—equivalently, $\mathbb{P}^2$—it takes the following form [Cox, 2.3]: "every definition remains significant, and every theorem remains true, when we interchange *point* and *line*, *join* and *intersection*." See [W, p. 155] for an analogous statement of the principle of triality.

Let ϕ be an automorphism of G, and let $\mathrm{SubSp}(V)$ denote the collection of subspaces of V. We want an efficient way to check if a given function

$\psi\colon \Gamma_V \to \mathrm{SubSp}(V)$ is the automorphism of the geometry Γ_V induced by ϕ.

10.1 Theorem. *If*

(1) $\psi(gX) = \phi(g)\psi(X)$ *for every* $X \in \Gamma_V$ *and* $g \in G$ *and*

(2) there is a chamber $\{V_i \mid 1 \le i \le n\}$ *such that* $\{\psi(V_i) \mid 1 \le i \le n\}$ *is also a chamber,*

then ψ *is the automorphism of the geometry* Γ_V *induced by the automorphism* ϕ *of* G*.*

[The term "chamber" was defined in 9.4.]

In the examples below, we will specify a function ψ and prove that it satisfies the hypotheses of the theorem. We will use all of the freedom implicit in our hypotheses on ψ. In the E_6 example, it will be clear that $\psi(X)$ is a subspace of V for $X \in \Gamma_V$, but it will not be obvious that $\psi(X)$ is in Γ_V. In the triality example, we will only define ψ on Γ_V and not for an arbitrary subspace of V. And in the type A example, ψ will not preserve the standard chamber.

Proof of Theorem 10.1. Let X be an object in Γ_V. We first claim that $\psi(X)$ is also in Γ_V. We find a chamber $\{X_i \mid 1 \le i \le n\}$ containing X such that X_i is of type α_i. This chamber is conjugate to the chamber $\{V_i \mid 1 \le i \le n\}$ from (2), i.e., there is some $g \in G$ such that $gV_i = X_i$ for every i. Therefore $\psi(X_i) = \phi(g)\psi(V_i)$, and $\psi(X_i)$ is an object in the geometry for all i.

Let P be the stabilizer of X in G. For $g \in \phi(P)$, we have

$$g\psi(X) = \psi(\phi^{-1}(g)X) = \psi(X) \quad \text{by (1)},$$

hence $\phi(P)$ is contained in the stabilizer of $\psi(X)$. But $\psi(X)$ is an object in Γ_V, hence by definition it is a nonzero, proper subspace of V. In particular, its stabilizer is a proper subgroup of G. Since P is a maximal proper subgroup of G, so is $\phi(P)$, hence $\phi(P)$ is the stabilizer of $\psi(X)$. This proves that the diagram

$$\begin{tikzcd}
\Gamma_P \arrow[r, "\phi"] & \Gamma_P \\
\Gamma_V \arrow[u] \arrow[r, "\psi"] & \Gamma_V \arrow[u]
\end{tikzcd}$$

commutes, where the vertical arrows send a subspace of V to its stabilizer in G. Since the vertical arrows are bijections (see §2), ψ is also a bijection. Moreover, ψ respects the notion of incidence in Γ_V, because that relation is the one transported from Γ_P by the vertical isomorphisms. We have proved that ψ is an automorphism of Γ_V, and the commutativity of the diagram shows that it is the one induced by ϕ. $\square$

10.2 Example (type A: projective duality). Let G be SL_n acting on k^n, and let ϕ be the automorphism $g \mapsto (g^{-1})^t$. For a subspace X of k^n, we define $\psi(X)$ to be the orthogonal complement of X with respect to the dot product defined by $x \cdot y := x^t\, y$.

Viewed algebraically, the dot product identifies k^n with the dual vector space $(k^n)^*$. This identification pairs $\psi(X)$ with the collection of linear forms vanishing on X.

Viewed geometrically, the map ψ is precisely the correspondence between points and hyperplanes giving projective duality in $\mathbb{P}^{n-1}$ described in [Pe] and [Cox, 11.8]. It interchanges a point $[a_1 : a_2 : \ldots : a_n]$ in homogeneous coordinates with the hyperplane consisting of solutions to the equation $\sum a_i x_i = 0$. For $n = 3$, it is the polarity with fundamental conic $x_1^2 + x_2^2 + x_3^2 = 0$, cf. [VY, §98].

We now check that ψ satisfies the conditions of Theorem 10.1. The dot product is compatible with the automorphism ϕ in the sense that

$$x \cdot y = (gx) \cdot (\phi(g)y) \quad \text{for } g \in SL_n \text{ and } x, y \in k^n.$$

Thus, a vector y is in $\psi(gX)$ if and only if $X \cdot (\phi(g)^{-1}y) = 0$, i.e., if and only if y is in $\phi(g)\psi(X)$. Thus ψ satisfies (1).

Consider the collection $\{V_1, V_2, \ldots, V_{n-1}\}$ of subspaces such that V_i consists of the vectors whose bottom $n-i$ coordinates are zero. Each V_i is stabilized by the upper triangular matrices—which make up a Borel subgroup—so this collection is a chamber. Applying ψ, we find that $\psi(V_i)$ is the space of vectors whose top $n-i$ coordinates are zero. Each $\psi(V_i)$ is stabilized by the lower triangular matrices. That is, $\{\psi(V_1), \psi(V_2), \ldots, \psi(V_{n-1})\}$ is also a chamber and ψ satisfies (2). Hence ψ is the automorphism of Γ_V corresponding to the automorphism ϕ of SL_n.

11 Example: type D (orthogonal duality)

Let G be a group as in §6, constructed from the root system of type D_n for some $n \geq 4$; it is traditionally denoted by Spin_{2n}. Its Dynkin diagram has an automorphism of order 2 given by interchanging the roots α_{n-1} and α_n. Here we will construct the corresponding automorphism ψ of the geometry Γ_V.

11.1. We can draw a Hasse diagram for the weights of V as we did for E_6 in Figure 7.2. The case $n = 4$ is shown in Figure 12.1 below. Figure 11.2 shows the diagram in the general case, rotated counterclockwise by 90° for space considerations.

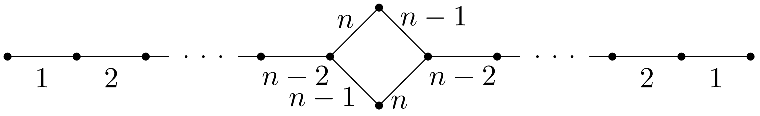


Figure 11.2: Hasse diagram of weights of the standard representation of D_n from [PSV, Fig. 4]. Larger weights are on the left.

Fix nonzero vectors $e_1, e_2, \ldots, e_n$ in V such that e_i has weight

$$\varepsilon_i := \omega_1 - \sum_{j=1}^{i-1} \alpha_j.$$

The longest element of the Weyl group is -1; it is the unique automorphism of the diagram of order 2 that stabilizes none of the weights. The weights ε_1 through ε_{n-1} are those in the string on the left of the diagram and ε_n is the bottom weight in the middle square. The other weights of V are of the form $-\varepsilon_i$ for some i. Let f_i be a nonzero vector of weight $-\varepsilon_i$, so the vectors $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n$ are a basis of V.

Let b be the G-invariant symmetric bilinear form on V as in §6. Clearly, since ε_i is not $-\varepsilon_j$ for any pair i, j, the subspace of V spanned by the e_i's (respectively, by the f_i's) is isotropic, i.e., $b(e_i, e_j) = b(f_i, f_j) = 0$ for all i, j. Also, $b(e_i, f_j)$ is nonzero if and only if $i = j$. By scaling the f_j's, we may assume that $b(e_i, f_j) = \delta_{ij}$ (Kronecker delta). (We have now obtained the description of G and $SO(b)$ given in [Br build, §V.7].) The construction in §6 gives:

$$\begin{aligned} V_{\alpha_i} &= k\text{-span}\,\{e_1, e_2, \ldots, e_i\} \text{ for } i \leq n-2, \\ V_{\alpha_{n-1}} &= k\text{-span}\,\{e_1, e_2, \ldots, e_n\}, \text{and} \\ V_{\alpha_n} &= k\text{-span}\,\{e_1, e_2, \ldots, e_{n-1}, f_n\}. \end{aligned} \tag{11.3}$$

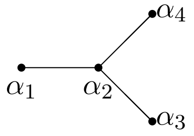
Let s denote the matrix in $GL(V)$ that fixes e_i and f_i for $1 \leq i < n$ and interchanges e_n and f_n. It leaves b invariant, and the map $\phi\colon SO(b) \to SO(b)$ defined by $\phi(g) = sgs^{-1}$ is an automorphism of order 2. There is a unique lift of ϕ to an automorphism of G [BT 72, 2.24(i)], which we also denote by

ϕ. The description of the root subgroups in $SO(b)$ in [Bor, 23.4] shows that ϕ is, in fact, the automorphism of G induced by the automorphism of the Dynkin diagram that interchanges α_{n-1} and α_n.

For each subspace X of V, put $\psi(X) := sX$. It is a triviality that ψ satisfies condition (1) of 10.1 and that the standard chamber exhibited in (11.3) is permuted by ψ, hence that ψ satisfies condition (2). That is, ψ is the automorphism of Γ_V induced by the automorphism ϕ of G and $SO(b)$.

12 Example: type D_4 (triality)

Continue the notation of the preceding section, §11, except suppose now that $n = 4$. The Dynkin diagram of G looks like



Let ϕ be the automorphism of order 3 that permutes the arms counterclockwise. We will now describe explicitly the corresponding automorphism ψ of the geometry Γ_V.

Let ρ_0 be the representation of G on V with highest weight ω_1. For $i = 1, 2$, we set $\rho_i := \rho_0\phi^{-i}$; it is a representation of G on V. The highest weight of ρ_1 is $\phi(\omega_1) = \omega_3$, and the highest weight of ρ_2 is $\phi^2(\omega_1) = \omega_4$.

The weights of V with respect to ρ_0 are listed in Figure 12.1; such a diagram is easily constructed as in 7.1. The weights relative to ρ_1 and ρ_2 are the same except with ϕ or ϕ^2 applied, respectively. Fix a basis e_i, f_j for V and a symmetric bilinear form b as in 11.1. The image $\rho_i(G)$ of G in $GL(V)$ is the same for all i, so b is $\rho_i(G)$-invariant for all i.

We claim that there is a nonzero linear map $t\colon V \otimes V \otimes V \to k$ that is G-invariant in the sense that

$$t(\rho_0(g)x_0, \rho_1(g)x_1, \rho_2(g)x_2) = t(x_0, x_1, x_2) \tag{12.2}$$

and is unique up to multiplication by a nonzero scalar. To see this, note that a linear map satisfying (12.2) is a G-fixed element of $\rho_0^* \otimes \rho_1^* \otimes \rho_2^*$, where ρ_i^* denotes the representation dual to ρ_i—which in this case is just ρ_i. We compute the formal character of this tensor product by multiplying the formal characters of the ρ_i [Hu 80, 22.5B], recover the decomposition of $\rho_0 \otimes \rho_1 \otimes \rho_2$ as a direct sum of irreducibles, and look for 1-dimensional summands

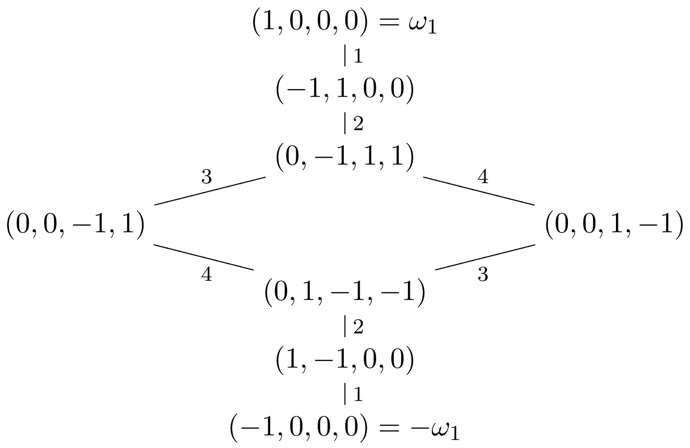


Figure 12.1: Hasse diagram of the weights of V relative to ρ_0.

as in 7.7. (The software package LiE uses the more efficient Brauer-Klimyk formula described in [Hu 80, Exercise 24.9].) We find a unique 1-dimensional summand; existence and uniqueness of t follows.

We can prove that t is nonzero for some specific arguments.

12.3 Lemma. *Let $x_i \in V$ be a nonzero vector of weight μ_i relative to ρ_i for $i = 0, 1, 2$. We have: $t(x_0, x_1, x_2)$ is nonzero if and only if $\mu_0 + \mu_1 + \mu_2 = 0$.*

Proof. "Only if" is clear, so we prove "if". Suppose that $\sum \mu_i = 0$. Since $\rho_2(G)$ acts transitively on the weights of V relative to ρ_2, we may assume that μ_2 is ω_4. By the argument in the proof of Lemma 7.11, the subgroup of the Weyl group fixing ω_4 has two orbits on the weights of V relative to ρ_1, with representatives $\pm\omega_3$. Since $-\omega_4 - \omega_3$ is not a weight of V relative to ρ_0, we must have $\mu_1 = -\omega_4 + \omega_3$. We have just proved that the Weyl group acts transitively on the triples μ_0, μ_1, μ_2 such that $\sum \mu_i = 0$. As in the proof of Lemma 7.13, it follows that $t(x_0, x_1, x_2)$ is nonzero. □

Moreover, t is invariant under cyclic permutations.

12.4 Lemma. *The value of t is unchanged if its arguments are permuted cyclically.*

Proof. Consider the linear map $d\colon V \otimes V \otimes V \to k$ defined by

$$d(x_0, x_1, x_2) := t(x_0, x_1, x_2) - t(x_1, x_2, x_0);$$

it is G-equivariant because $\rho_{i+1} = \rho_i \phi^{-1}$ for all i. By the uniqueness of t, the map d must be a scalar multiple of t. The vector $e_4 \in V$ is nonzero of weight $-\omega_3 + \omega_4$ relative to ρ_0. Then $t(e_4, e_4, e_4)$ is not zero by the previous lemma, yet $d(e_4, e_4, e_4)$ is zero. Therefore, d is identically zero. $\square$

We now define products $*_i$ on V for $i = 0, 1, 2$ implicitly via

$$t(x_0, x_1, x_2) = b(x_i, x_{i+1} *_i x_{i+2}).$$

By Lemma 12.4, all three products agree, so we write simply $*$. Because t and b are G-equivariant, in the sense that

$$(\rho_i(g)x) * (\rho_{i+1}(g)y) = \rho_{i+2}(g)(x * y). \tag{12.5}$$

This allows us to compute the multiplication, at least up to a scalar factor. Let $x_i \in V$ be nonzero with weight μ_i relative to ρ_i for $i = 1, 2$. It follows from Lemma 12.3 that $x_1 * x_2$ is nonzero if and only if $\mu_1 + \mu_2$ is a weight of V relative to ρ_0, in which case $x_1 * x_2$ has weight $\mu_1 + \mu_2$. We summarize these computations in the table below, where the entry in the row x_1 and column x_2 is "$\cdot$" if $x_1 * x_2$ is zero and, for example, e_3 if $x_1 * x_2$ is a nonzero scalar multiple of e_3. The left column lists the weight of x_1 for the reader's convenience; we omit the weight of x_2 due to space considerations. Since the product is G-equivariant and the weights of ρ_i are preserved under multiplication by -1, one needs only to compute the first four columns of entries; the remaining four columns can be filled in by symmetry.

			x_2							
			e_1	e_2	e_3	e_4	f_4	f_3	f_2	f_1
	$(0,0,1,0)$	e_1	$\cdot$	$\cdot$	$\cdot$	e_1	$\cdot$	e_2	e_3	f_4
	$(0,1,-1,0)$	e_2	$\cdot$	$\cdot$	e_1	$\cdot$	e_2	$\cdot$	e_4	f_3
	$(1,-1,0,1)$	e_3	$\cdot$	e_1	$\cdot$	$\cdot$	e_3	e_4	$\cdot$	f_2
x_1	$(1,0,0,-1)$	e_4	e_1	$\cdot$	$\cdot$	$\cdot$	f_4	f_3	f_2	$\cdot$
	$(-1,0,0,1)$	f_4	$\cdot$	e_2	e_3	e_4	$\cdot$	$\cdot$	$\cdot$	f_1
	$(-1,1,0,-1)$	f_3	e_2	$\cdot$	f_4	f_3	$\cdot$	$\cdot$	f_1	$\cdot$
	$(0,-1,1,0)$	f_2	e_3	f_4	$\cdot$	f_2	$\cdot$	f_1	$\cdot$	$\cdot$
	$(0,0,-1,0)$	f_1	e_4	f_3	f_2	$\cdot$	f_1	$\cdot$	$\cdot$	$\cdot$

(12.6)

Although the table above is not fine enough to allow us to actually multiply two vectors in V, it is sufficient to describe how the objects in the geometry Γ_V interact with the multiplication. Specifically, we can recover some of the results of [vdBS] without discussing octonion algebras.

12.7 Proposition. (Cf. [vdBS, §2])

(1) *If X is an α_1-space (a "point"), then $V * X$ is an α_3-space and $X * V$ is an α_4-space.*

(2) *If X is an α_2-space (a "line"), then $(X * V) * X$ and $X * (V * X)$ are α_2-spaces.*

(3) *If X is an α_3-space (resp., an α_4-space), then there is a unique α_1-space U such that $X = V * U$ (resp., $X = U * V$).*

Proof. By (12.5), it suffices to check (1) for the case where X is the k-span of e_1, i.e., V_{α_1}. In that case, (1) is clear from the multiplication table. A similar argument handles (2).

We now prove (3) for α_3-spaces. As in the previous paragraph, it suffices to check the case where X is the k-span of e_1, e_2, e_3, e_4, i.e., V_{α_3}. The multiplication table shows that X is $V * e_1$, so suppose that $u \in V$ is nonzero and satisfies $V * u = X$. Considering the product $f_1 * u \in X$, we see from the multiplication table that the coefficients of e_2, e_3, and f_4 in u are all zero. Replacing $f_1 * u$ with $f_2 * u$, etc., we conclude that u is in ke_1. $\square$

Define ψ via

$$\psi(ka) = a * V, \quad \psi(a * V) = V * a, \quad \text{and} \quad \psi(V * a) = ka \tag{12.8}$$

for a isotropic in V. This is well defined by the proposition. For X an α_2-space, we define

$$\psi(X) = X * (V * X). \tag{12.9}$$

Applying (12.5), it is easy to check that ψ satisfies condition (1) of Th. 10.1. On the other hand, we checked in the proof of Prop. 12.7 that ψ maps the standard chamber to the standard chamber, so ψ also satisfies condition (2). Thus ψ is the automorphism of Γ_V corresponding to ϕ.

Remarks. Equation (12.8) defines ψ^2 on the α_1-, α_3-, and α_4-spaces. Appealing to Th. 10.1, one finds that $\psi^2(X)$ is $(X * V) * X$ for X an α_2-space.

We remark that we have recovered a multiplication of the octonions—at least approximately—entirely from first principles of representation theory.

13 Example: type E_6 (duality)

Let G be the split simply connected group of type E_6 with standard representation V as in §7. In this section, we will give an explicit description of the automorphism ψ of the geometry Γ_V corresponding to the automorphism ϕ of the group G.

We define the *brace product* on V following [McC book, p. 190]:

$$\{x, y, z\} := b(x, y)z + b(z, y)x - (x\#z)\#y.$$

Note that x and z are interchangeable. (See Remark 13.21 for comments on why we choose to work with the brace product.)

For each subspace X of V, we set

$$\boxed{\psi(X) := \{y \in V \mid \{X, y, V\} \subseteq X\}} \tag{13.1}$$

From (7.9), we find that

$$g\{x, y, z\} = \{gx, \phi(g)y, gz\} \quad \text{for } g \in G \text{ and } x, y, z \in V. \tag{13.2}$$

An argument nearly identical to the one in Example 10.2 shows that ψ satisfies hypothesis (1) of Th. 10.1. The rest of this section is spent proving that $\psi(V_\delta) = V_{\phi(\delta)}$ for all $\delta \in \Delta$, i.e., ψ permutes the objects in the standard chamber. This will show that ψ satisfies hypothesis (2) of the theorem.

13.3 Example. Let U denote the set of weights μ of V such that $\phi(\omega_1 + \mu)$ is *not* a weight of V. Let z be a nonzero vector of weight $\mu \in U$; we claim that z is in $\{v^+, v^-, V\}$, where v^- is a lowest weight vector of V. First observe that since z does not have weight $-\omega_6$, $b(v^+, z)$ is zero. Since μ is in U, $v^+\#z$ is zero and we have: $\{v^+, v^-, z\} = b(v^+, v^-)z$. But $b(v^+, v^-)$ is not zero because b is nondegenerate. This proves the claim.

The image of the map $x \mapsto v^+\#x$ is 10-dimensional, so the subspace of V spanned by weight vectors with weights in U is 17-dimensional. Therefore,

$$\dim\{v^+, v^-, V\} \geq 17.$$

13.4. Connection with Jordan theory, part II. Let N and b be as in 7.18. That is, we suppose for the moment that k is algebraically closed and view N and b as the generic norm and trace bilinear form on an Albert algebra, respectively. We will use three facts from the theory of Albert algebras. The first is the *5-linear identity* from [McC book, p. 202]:

$$\{x, y, \{z, w, u\}\} = \{\{x, y, z\}, w, u\} - \{z, \{y, x, w\}, u\} + \{z, w, \{x, y, u\}\}.$$

Second, we will use the classification of the inner ideals of V. A subspace X is an *inner ideal* if $\{X, V, X\}$ is contained in X. By [McC 71, §7], the proper inner ideals are the singular subspaces—which all have dimension ≤ 6—and the hyperlines. The α_2- and α_6-spaces are *maximal* proper inner ideals. A straightforward application of the 5-linear identity gives: *If I is an inner ideal in V, then $\psi(I)$ is also an inner ideal.*

Finally, we will use the *adjoint identity*:

$$x^{\#\#} = N(x)x \quad \text{for } x \in V. \tag{13.5}$$

(Of course, this identity can also be proved using the representation-theoretic fact that there is a unique G-invariant line in $V \otimes S^4(V^*)$.)

We claim that the three facts hold also in the case where k is not algebraically closed. For the adjoint identity, this is clear. Similarly, a subspace X of V is an inner ideal if and only if $X \otimes \overline{k}$ is an inner ideal in $V \otimes \overline{k}$. The only potentially tricky check is the claim that every nonsingular, nonzero, and proper inner ideal X is a hyperline. To see this, note that $X \otimes \overline{k}$ is a hyperline by the case where k is algebraically closed, so $X \otimes \overline{k}$ equals $(X \otimes \overline{k})^{\#} \# (V \otimes \overline{k})$ by Remark 7.16. But we may rewrite this as $(X^{\#} \# V) \otimes \overline{k}$. Since $(X \otimes \overline{k})^{\#}$ is 1-dimensional and singular, so is $X^{\#}$. This shows that X is the hyperline $X^{\#} \# V$.

13.6. Computation of $\psi(V_{\alpha_1})$. Linearizing the adjoint identity as in [McC 69, p. 496], we find the identity (McCrimmon's Equation (12)):

$$(x\#y)\#(x\#z) = b(x^{\#}, y)z + b(x^{\#}, z)y + b(y\#z, x)x - x^{\#}\#(y\#z) \tag{13.7}$$

For the highest weight vector v^+, we have $(v^+)^{\#} = 0$ and (13.7) becomes

$$(v^+\#y)\#(v^+\#z) = b(y\#z, v^+)v^+.$$

We find:

$$\{v^+, v^+\#z, y\} = b(v^+, v^+\#z)y + b(y, v^+\#z)v^+ - b(y\#z, v^+)v^+.$$

Since $N(-,-,-)$ is symmetric, equation (7.8) shows that the first summand is zero and the second and third summands cancel. Therefore, $\{v^+, v^+\#z, y\}$ is zero and $\psi(kv^+)$ contains $v^+\#V$.

Since $\psi(kv^+)$ is an inner ideal containing the hyperline $v^+\#V$ and it is proper (Example 13.3), the ideal must be the hyperline. Since G acts transitively on the singular 1-dimensional subspaces and ψ satisfies 10.1.1, we obtain the following lemma:

13.8 Lemma. *If X is a 1-dimensional singular subspace (= an α_1-space), then $\psi(X)$ is the hyperline $X\#V$.*

13.9. **Computation of $\psi(V_{\alpha_2})$.** We now show that $\psi(V_{\alpha_2})$ is V_{α_2}. For x, y weight vectors in V_{α_2} and w a weight vector in V, we find that $\{x, y, w\}$ has weight at least

$$\lambda_2 + \phi(\lambda_2) - \omega_6 = \omega_1 - 2(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6).$$

But every weight of V of the form $\omega_1 - (c_1\alpha_1 + c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5 + c_6\alpha_6)$ with each c_i a nonnegative integer belongs to V_{α_2} by Lemma 3.2. Hence V_{α_2} is contained in $\psi(V_{\alpha_2})$.

Since v^- is not in $\psi(V_{\alpha_2})$ by Example 13.3 and $\psi(V_{\alpha_2})$ is an inner ideal, the classification of inner ideals gives that $\psi(V_{\alpha_2})$ is precisely V_{α_2}.

13.10 Example. Let y be a nonzero vector of weight λ_2, i.e., a lowest weight vector for L_{α_2} acting on V_{α_2}. We claim that $\{v^+, y, V\}$ is V_{α_2}. Since y is in $\psi(V_{\alpha_2})$ by 13.9, we need only show that V_{α_2} is contained in $\{v^+, y, V\}$. It suffices to prove that every weight of V_{α_2} is a weight of $\{v^+, y, V\}$ because every weight of V has multiplicity one.

Consulting Figure 7.2, we see that the weights of V_{α_2} are symmetric in the following sense: If $\omega_1 - \alpha$ is a weight of V_{α_2}, then $\lambda_2 + \phi(\alpha)$ is also a weight of V_{α_2}. Consequently, for every weight λ of V_{α_2},

$$f(\lambda) := -\phi(\lambda_2 + \phi(\omega_1 - \lambda)) = -\phi(\lambda_2) + \lambda - \omega_1$$

is a weight of V. For each weight λ of V_{α_2}, fix a nonzero vector z_λ of weight $f(\lambda)$. The vector $\{v^+, y, z_\lambda\}$ has weight λ, and it suffices to prove that it is not zero for each λ. Note that $b(v^+, y)$ is zero because $\omega_1 + \phi(\lambda_2)$ is not zero.

For $\lambda = \omega_1$, we note that $\omega_1 + f(\lambda) = \omega_1 - \phi(\lambda_2)$ has 2 as one of its entries, hence $\phi(\omega_1 + f(\lambda))$ is not a weight of V. Therefore, $v^+ \# z_\lambda$ is zero. We have

$$\{v^+, y, z_\lambda\} = b(z_\lambda, y)v^+,$$

which is not zero because $f(\lambda) = -\phi(\lambda_2)$.

For the other five weights λ of V_{α_2}, note that $f(\lambda) \neq -\phi(\lambda_2)$, so $b(z_\lambda, y)$ is zero. We claim that $v^+ \# z_\lambda$ is not zero. It has weight $\mu := \phi(\omega_1 + f(\lambda)) = \phi(\lambda) - \lambda_2$. For $\lambda = \lambda_3$, Equation (7.14) gives that $\mu = \lambda_5$, a weight of V. For $\lambda = \lambda_4 = \lambda_3 - \alpha_3$, we find that μ is $\lambda_5 - \alpha_5$. Similarly, we find that for each of the the three remaining λ's, the weight μ is a weight of V. That is, $v^+ \# z_\lambda$ is nonzero. The function f was defined so that $(v^+ \# z_\lambda) \# y$ would have weight λ, hence that product is also not zero, i.e.,

$$\{v^+, y, z_\lambda\} = (v^+ \# z_\lambda) \# y \neq 0.$$

We have proved that $\{v^+, y, V\}$ is V_{α_2}.

We can now give a reasonably good description of the space $\{x, y, V\}$ when x and y are nonzero weight vectors. Note that x and y are necessarily singular because no weight of V has 2 as a coordinate. We say that y and $x\#V$ are *connected* if there is a singular vector $z \in x\#V$ such that y and z are "collinear", i.e., such that y and z span an α_3-space. (In this case, Tits says that y and $x\#V$ are "liés" in [T 57, 3.9].) For example, the vector y from Example 13.10 and $v^+\#V$ are connected because y and v^+ are in the α_2-space V_{α_2}. In contrast, the lowest weight vector v^- and $v^+\#V$ are not connected, as $\phi(-\omega_6 + \mu)$ is a weight of V for every weight μ of $v^+\#V$. We find:

13.11 Lemma (Cf. [SV 68, 3.16]). *For x and y nonzero weight vectors in V,*

$$\dim\{x, y, V\} \begin{cases} = 0 & \text{if } y \text{ and } x\#V \text{ are incident} \\ = 6 & \text{if } y \text{ and } x\#V \text{ are not incident but are connected} \\ \geq 17 & \text{if } y \text{ and } x\#V \text{ are neither incident nor connected.} \end{cases}$$

When the dimension is 6, $\{x, y, V\}$ *is an* α_2*-space.*

Proof. First consider the case where x is the highest weight vector v^+. We remind the reader that y and $v^+\#V$ are incident if and only if y is contained in $v^+\#V$, so the first equality follows from 13.6. By Lemma 7.11, we are reduced to considering the cases where y has weight λ_2 or $-\omega_6$. The second equality and the inequality now follow by Examples 13.10 and 13.3, respectively.

In the general case, since the Weyl group acts transitively on the weights of V, there is some $n \in G$ such that nx is a nonzero multiple of v^+. The dimension of $\{x, y, V\}$ and the properties of y and $x\#V$ being incident or connected are not affected by replacing x and y with nx and $\phi(n)y$, respectively. The general case now follows from the previous paragraph. $\square$

13.12 Remark. Since G acts "strongly transitively" on the geometry Γ_V (see [T 74, 3.2.6]), it is sufficient to only consider the case where x and y are weight vectors. That is, (13.11) holds for *every* pair of singular vectors x, y. We will not use this fact.

13.13 Lemma. *Fix a proper inner ideal X of V such that* $\dim X \neq 6$. *If X has a basis $\mathcal{B}$ consisting of weight vectors, then*

$$\psi(X) = \bigcap\nolimits_{b \in \mathcal{B}} b\#V$$

and $\{X, \psi(X), V\} = 0$.

Proof. Since X has a basis consisting of weight vectors, so does $\psi(X)$. Fix a $b \in \mathcal{B}$ and a weight vector $y \in \psi(X)$. We claim that $\{b, y, V\}$ is the zero subspace. Otherwise, by Lemma 13.11, $\{b, y, V\}$ is an α_2-space or has dimension at least 17. In particular, the dimension of X is at least 7. This implies that X is a hyperline, so it cannot contain an α_2-space by Remark 7.16, and we have a contradiction. This proves that $\{b, y, V\}$ is zero. Letting y vary, we find that $\{b, \psi(X), V\}$ is zero, which proves the second equation. By Lemma 13.8, $\psi(X)$ is contained in $b\#V$.

Conversely, suppose that y is in the intersection of the $b\#V$'s. Then $\{x, y, V\}$ is zero for every $x \in X$. Since the b's span X, y is in $\psi(X)$. This proves the displayed equation. $\square$

We can now compute $\psi(V_{\alpha_i})$ for $i \neq 1, 2$.

13.14. Computation of V_{α_3} and V_{α_4}. The space V_{α_3} is spanned by the highest weight vector v^+ and a vector x of weight λ_3. We wish to compute $\psi(V_{\alpha_3})$, which is $V_{\alpha_6} \cap (x\#V)$ by Lemma 13.13. Each weight τ of V_{α_6} is a weight of $x\#V$ if and only if $\phi(\tau) - \lambda_3$ is a weight of V. The five weights τ of V_{α_6} with a 1 as their last coordinate cannot belong to $x\#V$ because $\phi(\tau) - \lambda_3$ has a 2 as its first coordinate. That is, $\psi(V_{\alpha_3})$ is contained in V_{α_5}.

Since $\phi(\lambda_5) - \lambda_3 = -\phi(\lambda_2)$ is a weight of V, the weight λ_5 belongs to $x\#V$. Figure 7.2 shows that $-\phi(\lambda_2) + \alpha_2$ is also a weight of V, hence $\lambda_5 + \alpha_2$ belongs to $x\#V$. Continuing in this manner, we find that V_{α_5} is contained in $x\#V$, hence that $\psi(V_{\alpha_3})$ is V_{α_5}.

The space V_{α_4} is spanned by V_{α_3} and a vector y of weight λ_4. The two weights τ of V_{α_5} that do not belong to V_{α_4} each have a 1 as their 5th coordinate, hence $\phi(\tau) - \lambda_4$ has a 2 as its 3rd coordinate, and such a τ does not belong to $\psi(V_{\alpha_4})$. The three weights of V_{α_4} are easily checked to be weights of $y\#V$, hence $\psi(V_{\alpha_4})$ is V_{α_4}.

13.15. Computation of V_{α_5} and V_{α_6}. By Lemma 13.13 and 13.14, $\psi(V_{\alpha_5})$ is contained in V_{α_4}. Moreover, the 5th coordinate of $\phi(\lambda_4) - \lambda_5$ is -2, hence $\psi(V_{\alpha_5})$ is contained in V_{α_3}.

Equation (7.14) gives that the weight λ_3 belongs to $z\#V$ for z a nonzero vector of weight λ_5. Also,

$$\lambda_1 = \lambda_3 + \alpha_1 = \phi(\lambda_5 + (\lambda_2 + \alpha_6)),$$

so v^+ belongs to $z\#V$. Similar calculations show that V_{α_3} is contained in $x\#V$ for x of weight $\lambda_5 + \alpha_2$, hence V_{α_3} is equal to $\psi(V_{\alpha_5})$.

Lemma 13.13 gives that $\psi(V_{\alpha_6})$ is contained in the 2-dimensional space $\psi(V_{\alpha_5}) = V_{\alpha_3}$. The 6th coordinate of $\phi(\lambda_3) - \lambda_6$ is -2, hence λ_3 does not belong to $u\#V$ for every vector u of weight λ_6, and $\psi(V_{\alpha_6})$ is contained in the k-span of the highest weight vector v^+.

We now show that v^+ is in $\psi(V_{\alpha_6})$. Linearizing the identity $x^{\#\#} = N(x)x$ as in [McC 69] again (and going through McCrimmon's Equation (19)), we find the identity

$$z\#(y\#(x\#z)) = b(x, z^{\#})y + b(x, y)z^{\#} + b(y, z)(x\#z) - x\#(y\#z^{\#}), \quad (13.16)$$

which holds for every $x, y, z \in V$. Substituting $z \mapsto v^+$, we find

$$v^+\#(y\#(x\#v^+)) = b(y, v^+)(x\#v^+).$$

Recalling that $b(v^+\#x, v^+)$ is zero, we find that $\{v^+\#x, v^+, y\}$ is zero for all $x, y \in V$. That is, v^+ is in $\psi(V_{\alpha_6})$ and $\psi(V_{\alpha_6})$ is V_{α_1}.

We have proved that $\psi(V_{\alpha_i}) = V_{\phi(\alpha_i)}$ for all i. In particular, the image of the standard chamber under ψ is just the standard chamber. This proves that ψ is the automorphism of Γ_V induced by the automorphism ϕ of G.

In the language of classical projective geometry, ψ is a *hermitian polarity.* Indeed, since ϕ^2 is the identity on G, ψ^2 is the identity on Γ_V, i.e., ψ is a polarity. One says that ψ is hermitian because the "point" V_{α_1} is contained in its "polar" V_{α_6}.

It is natural to ask what ψ does to inner ideals that are not in Γ_V, namely the 4-dimensional singular and 5-dimensional non-maximal singular subspaces.

13.17 Example. In [T 57, 2.3], Tits defines a geometry called an "R-espace". It contains our geometry Γ_V plus two other types of objects: the 4- and 5-dimensional subspaces found as intersections in 7.21. (Algebraically, the objects of Tits's R-espace are the nonzero, proper inner ideals in an Albert algebra.) We now compute ψ on these "extra" subspaces.

Let X be the intersection of V_{α_2} and V_{α_5}. It is spanned by V_{α_4} and a nonzero vector x of weight $(0, 1, 0, -1, 1, 0)$. By Lemma 13.13, we have:

$$\psi(X) = \psi(V_{\alpha_4}) \cap (x\#V) = V_{\alpha_3}$$

Note that the 4-dimensional X is properly contained in the 5-dimensional $\psi(\psi(X)) = V_{\alpha_5}$. These results are what one expects from Tits's perspective [T 57, 3.3]. But algebraically they are in contrast with V_δ for $\delta \in \Delta$, for which we have $\psi(\psi(V_\delta)) = V_\delta$.

The subspace Y from 7.21—spanned by X and a vector with weight $(0, 1, 0, 0, -1, 1)$—is similar. The analogous computation shows that $\psi(Y)$ is V_{α_1}, hence $\psi(\psi(Y))$ is the hyperline V_{α_6}.

Combining Lemma 13.13 with the fact that every object in Γ_V is in the G-orbit of V_δ for some δ, we obtain the following corollary:

13.18 Corollary. *If $X \in \Gamma_V$ is not of dimension 6, then*

$$\psi(X) = \{z \in V \mid \{X, z, V\} = 0\}.$$

Faulkner discussed the geometry Γ_V in terms of the brace product in [Fa], although he focussed on the points (α_1-spaces) and hyperlines. He described the duality on points and hyperlines by the equation displayed in the corollary. However, that definition does not work for our purposes. To see this, note that the the set

$$\{z \in V \mid \{V_{\alpha_2}, z, V\} = 0\}$$

is the zero space by computations as in Example 13.17. But the polar of V_{α_2} must be a 6-dimensional inner ideal.

13.19 Proposition. *For $X \in \Gamma_V$, we have*

$$b(X, \psi(X)) = 0 \quad \textit{and} \quad \{\psi(X), X, \psi(X)\} = 0.$$

Proof. By the transitivity of the G-action, we may assume that X is V_{α_i} for some i. Let j be such that $\alpha_j = \phi(\alpha_i)$. Further, let d_i be such that $\omega_1 - d_i = \lambda_i$; it is a sum of positive roots.

We first argue that $b(X, \psi(X))$ is zero. By (7.6), it can only be nonzero if there are vectors $x \in X$ and $x' \in \psi(X)$ of weights λ and λ' such that $\lambda + \phi(\lambda') = 0$. But every weight of X (resp. $\psi(X)$) is at least λ_i (resp. λ_j), and

$$\lambda_i + \phi(\lambda_j) = (\omega_1 + \omega_6) - (d_i + \phi(d_j));$$

we will show that this is > 0 for all i. Consulting the tables in [Bou Lie], we find:

$$\omega_1 + \omega_6 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6.$$

When $i = 1$, d_1 is zero and d_6 is a sum of positive roots with no root occurring more than twice, as can be seen from Figure 7.2. Therefore, $\lambda_1 + \phi(\lambda_6) > 0$. Applying ϕ to both sides of the equation, we find that $\lambda_6 + \phi(\lambda_1) > 0$. When $i = 2, 3, 4$, or 5, no root appears more than once in d_i. But $2 \leq j \leq 5$, hence the same is also true of d_j. Therefore no root appears in $d_i + \phi(d_j)$ more than twice. We have proved that $\lambda_i + \phi(\lambda_j) > 0$ for all i, hence $b(X, \psi(X))$ is necessarily zero.

We now prove that the second equation holds. The space $\{\psi(X), X, \psi(X)\}$ is a direct sum of its weight spaces, and each weight μ is at least $\phi(\lambda_i)+2\lambda_j$ by (13.2). That is, each weight μ is of the form $\omega_1 - d$ for

$$0 \leq d \leq \omega_1 - (\phi(\lambda_i) + 2\lambda_j) = (\phi(d_i) + 2d_j) - (\omega_1 + \omega_6). \tag{13.20}$$

It follows from the definition of λ_j as the lowest weight of V_{α_j} that d_j does not include the root α_j. Similarly, $\phi(d_i)$ also does not include α_j. Therefore, the coefficient of α_j on the right side of (13.20) is negative. In particular, the equation $d \geq 0$ is impossible, and μ cannot be a weight of V. This proves that $\{\psi(X), X, \psi(X)\}$ is zero. □

In [LN, p. 260], Loos and Neher defined

$$\mathrm{Inid}(X) := \{y \in V \mid \{y, X, y\} = 0 \text{ and } \{X, y, V\} \subseteq X\},$$

for X a subspace of V. Clearly, $\psi(X)$ contains $\mathrm{Inid}(X)$, and the preceding proposition shows that the two concepts agree for $X \in \Gamma_V$.

13.21 Remark. We used the brace product in this section so that we could apply results from Jordan theory; for an explanation of why Jordan theorists like it, see [McC book, pp. 7ff]. There is another ternary product that is perhaps more natural from the perspective of representation theory. Up to a scalar multiple, there is a unique G-invariant linear map $\langle\rangle$ from $V \otimes V^*$ to the Lie algebra of G. The composition

$$[x, y, z] := \langle x, h(y)\rangle\, z$$

satisfies an equation analogous to (13.2). Moreover, if one choose the appropriate multiple of $\langle\rangle$, then

$$[x, y, z] = \{x, y, z\} - \frac{2}{3} b(x, y) z \quad \text{for } x, y, z \in V.$$

(This product was studied in [Fr, 26.9ff] and [SV 68].) A subspace of V is an inner ideal with respect to the brace product if and only if it is an inner ideal with respect to the bracket product. If we define $[\psi](X)$ to be the set of all $y \in V$ such that $[X, y, V] \subseteq X$, then $\psi(X)$ is contained in $[\psi](X)$ for $X \in \Gamma_V$ by Prop. 13.19. In fact, ψ and $[\psi]$ agree for α_i-spaces with $i \neq 2$ by [Ga 01, 5.4, 6.7].

We remark that the map $z \mapsto \{x, y, z\}$ belongs to the Lie algebra of the *structure group* of V. That group is a reductive envelope of G with center a rank 1 torus.

13.22 Remark. A Jordan theorist would start not with a split group of type E_6 but rather with a "cubic norm structure" (a characteristic-free version of an Albert algebra) consisting of the 27-dimensional vector space V, the map $x \mapsto x^\#$, the cubic form $N : V \to k$, and a base point $1 \in V$, all satisfying certain axioms [McC 69, §1]. From that perspective the results of this paper hold without the restriction that k has characteristic zero. First, the subgroup of $GL(V)$ of elements that preserve N form a split simply connected group of type E_6 [Sp 73, p. 151]. Sections 1 through 3 go through with no change, see Remark 5.5. For §7, there is no need to construct N and $\#$. The form b is obtained as a logarithmic derivative of N, see e.g. [Sp 73, 1.18]. The material from 7.10 through the end of §7 goes through with only cosmetic changes. All of §13 goes through when the characteristic of k is different from 2. (To get theorems that work in all characteristics, one should consider not the brace product but rather a quadratic map whose bilinearization is the brace product. We have not done so here in order to make the exposition somewhat smoother.)

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References

[A E6] M. Aschbacher, *The 27-dimensional module for E_6. I*, Invent. Math. **89** (1987), 159–195.

[A G2] ______, *Chevalley groups of type G_2 as the group of a trilinear form*, J. Algebra **109** (1987), 193–259.

[BCP] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system I: the user language*, J. Symbolic Comput. **9** (1990), 677-698.

[Bor] A. Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[Bou Alg] N. Bourbaki, *Algebra II: Chapters 4–7*, Springer-Verlag, Berlin, 1988.

[Bou Lie] ______, *Lie groups and Lie algebras: Chapters 4–6* and *Chapters 7–9*, Springer-Verlag, Berlin, 2002 and 2005.

[Br build] K. Brown, *Buildings*, Springer, New York-Berlin, 1989.

[Br E7] R.B. Brown, *Groups of type E_7*, J. reine angew. Math. **236** (1969), 79–102.

[BT 72] A. Borel and J. Tits, *Compléments à l'article: Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 253–276.

[Ca th] E. Cartan, *Sur la structure des groupes de transformations finis et continus*, thesis, Paris, 1894; 2nd edition, Vuibert, 1933 (= Oe., part 1, vol. 1 (1952), 137–287).

[Ca 25] ———, *Le principe de dualité et la théorie des groupes simples et semi-simples*, Bull. Sc. Math. **49** (1925), 361–374 (= Oe., part 1, vol. 1 (1952), 555–568).

[Ch] C. Chevalley, *The algebraic theory of spinors*, reprint of the 1954 edition in *The algebraic theory of spinors and Clifford algebras*, Springer, Berlin, 1997.

[Coh] A.M. Cohen, *Point-line spaces related to buildings*, Handbook of incidence geometry, North-Holland, Amsterdam, 1995, pp. 647–737.

[Coop] B.N. Cooperstein, *The fifty-six-dimensional module for E_7, I. A four form for E_7*, J. Algebra **173** (1995), 361–389.

[Cox] H.S.M. Coxeter, *The real projective plane*, 2nd., Cambridge University Press, New York, 1960.

[Di] L.E. Dickson, *The configurations of the 27 lines on a cubic surface and the 28 bitangents to a quartic curve*, Amer. Math. Soc. Bull. (2) **8** (1901), 63–70 (= Coll. Math. Papers, vol. V, #181).

[Dy] E.B. Dynkin, *Maximal subgroups of the classical groups*, Amer. Math. Soc. Transl., series 2, vol. 6 (1957), 245–378 (= Selected Papers, pp. 37–170).

[Fa] J.R. Faulkner, *Octonion planes defined by quadratic Jordan algebras*, Mem. Amer. Math. Soc. (1970), no. 104.

[Fe] J.C. Ferrar, *Strictly regular elements in Freudenthal triple systems*, Trans. Amer. Math. Soc. **174** (1972), 313–331.

[FF] J.R. Faulkner and J.C. Ferrar, *Exceptional Lie algebras and related algebraic and geometric structures*, Bull. London Math. Soc. **9** (1977), no. 1, 1–35.

[FH] W. Fulton and J. Harris, *Representation theory: a first course*, Graduate texts in mathematics, vol. 129, Springer, 1991.

[Fr] H. Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene. VIII, IX*, Nederl. Akad. Wetensch. Proc. Ser. A **62** (1959), 447–474.

[Ga 99] R.S. Garibaldi, *Twisted flag varieties of trialitarian groups*, Comm. Algebra **27** (1999), no. 2, 841–856.

[Ga 01] ———, *Structurable algebras and groups of type E_6 and E_7*, J. Algebra **236** (2001), no. 2, 651–691, [DOI `10.1006/jabr.2000.8514`].

[GW] R. Goodman and N. Wallach, *Representations and invariants of the classical groups*, Cambridge University Press, 1998.

[Hu 80] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, 1980, Third printing, revised.

[Hu 90] ———, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, 1990.

[IS] V.A. Iskovskikh and I.R. Shafarevich, *Algebraic surfaces*, in “Algebraic Geometry II”, Encyclopaedia Math. Sci. **35**, Springer-Verlag, Berlin, 1996.

[Jac 3] N. Jacobson, *Some groups of transformations defined by Jordan algebras III*, J. Reine Angew. Math. **207** (1961), 61–85 (= Coll. Math. Papers, vol. 2, #65).

[KMRT] M.-A. Knus, A.S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Colloquium Publications **44**, Amer. Math. Soc., 1998.

[Kr] S. Krutelevich, *Jordan algebras, exceptional groups, and higher composition laws*, `arxiv:math.NT/0411104`.

[Lam] T.Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics **67**, Amer. Math. Soc., 2004.

[LN] O. Loos and E. Neher, *Complementation of inner ideals in Jordan pairs*, J. Algebra **166** (1994), 255–295.

[Lu] J. Lurie, *On simply laced Lie algebras and their minuscule representations*, Comment. Math. Helv. **76** (2001), 515–575.

[Ma] Yu. I. Manin, *Cubic forms*, translated from the Russian by M. Hazewinkel, second edition, North-Holland, Amsterdam, 1986.

[McC 69] K. McCrimmon, *The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras*, Trans. Amer. Math. Soc. **139** (1969), 495–510.

[McC 71] ———, *Inner ideals in quadratic Jordan algebras*, Trans. Amer. Math. Soc. **159** (1971), 445–468.

[McC book] ———, *A taste of Jordan algebras*, Universitext, Springer-Verlag, New York, 2004.

[MPW] A.S. Merkurjev, I.A. Panin, and A.R. Wadsworth, *Index reduction formulas for twisted flag varieties II*, K-theory **14** (1998), 101–196.

[N] E. Neher, *Jordan triple systems by the grid approach*, Lecture Notes in Mathematics **1280**, Springer-Verlag, Berlin, 1987.

[OS] P. Orlik and L. Solomon, *Singularities II: automorphisms of forms*, Math. Ann. **231** (1978), 229–240.

[Pe] D. Pedoe, *Geometry*, second ed., Dover, New York, 1988.

[PSV] E. Plotkin, A. Semenov, and N. Vavilov, *Visual basic representations: an atlas*, Internat. J. Algebra Comput. **8** (1998), no. 1, 61–95.

[Ra] M.L. Racine, *Point spaces in exceptional quadratic Jordan algebras*, J. Algebra **46** (1977), 22–36.

[Sc] R. Scharlau, *Buildings*, in “The handbook of incidence geometry”, North-Holland, Amsterdam, 1995, pp. 477–645.

[Sp 73] T.A. Springer, *Jordan algebras and algebraic groups*, Springer-Verlag, New York, 1973.

[Sp 98] ———, *Linear algebraic groups*, second ed., Birkhäuser, 1998.

[St lect] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968.

[St mem] ———, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc., no. 80, 1968.

[SV 68] T.A. Springer and F.D. Veldkamp, *On Hjelmslev-Moufang planes*, Math. Z. **107** (1968), 249–263.

[SV 00] ———, *Octonions, Jordan algebras and exceptional groups*, Springer-Verlag, Berlin, 2000.

[T 56] J. Tits, *Les groupes de Lie exceptionnels et leur interprétation géométrique*, Bull. Soc. Math. Belg. **8** (1956), 48–81.

[T 57] ———, *Sur la géométrie des R-espaces*, J. Math. Pures Appl. **36** (1957), 17–38.

[T 74] ———, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, vol. 386, Springer, 1974.

[TW] J. Tits and R.M. Weiss, *Moufang polygons*, Springer-Verlag, Berlin, 2002.

[Va] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Graduate Texts in Mathematics, vol. 102, Springer-Verlag, New York, 1984.

[vdBS] F. van der Blij and T.A. Springer, *Octaves and triality*, Nieuw Arch. Wisk. (3) **8** (1960), 158–169.

[vLCL] M.A.A. van Leeuwen, A.M. Cohen, and B. Lisser, *LiE, a package for Lie group computations*, Computer Algebra Nederland, Amsterdam, 1992.

[VY] O. Veblen and J.W. Young, *Projective geometry*, vol. 1, Ginn and Co., 1910.

[W] E.A. Weiss, *Punktreihengeometrie*, Teubner, Leipzig, 1939.