DEGREE 5 INVARIANT OF E_8

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Abstract. We give a formula for the recently-discovered degree 5 cohomological invariant of groups of type E_8 and use this formula to give a precise interpretation of Serre's "funny-looking statement" in terms of embeddings finite subgroups in the split E_8.

1. Introduction

Let G be a split simple linear algebraic group over a field k of characteristic 0. One of the main goals of the theory of linear algebraic groups over arbitrary fields is to compute the Galois cohomology set $H^1(k, G)$.

One of the main tools was suggested by J-P. Serre in the 90s, namely the *Rost invariant*

$$r_G \colon H^1(*, G) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2))$$

discovered by M. Rost and explained in Merkurjev's portion of the book [GMS]. It is a morphism of functors from the category of fields over k to the category of pointed sets.

Mimicking the situation in topology one can consider the kernel of the Rost invariant and try to define a cohomological invariant on it. In the theory of quadratic forms this procedure leads to the invariants defined on the powers of the fundamental ideal I^n.

In the present paper we consider the most complicated and yet unsettled case when G has Cartan-Killing type E_8. The paper is organized as follows. In Section 2 we recall the recently-discovered invariant u defined on the kernel of the Rost invariant for groups of type E_8. Section 3 is devoted to a computation of the value $u(G)$ for groups G obtained by a Tits construction. We also provide applications of u to cohomological invariants and essential dimension of Spin_{16}. In the last section we investigate the finite subgroups of algebraic groups. It turns out that under some additional conditions cohomological invariants provide an obstruction for certain finite groups to be subgroups of algebraic groups. This is connected with Serre's "funny-looking statement" from [GR, p. 209]:

> "Let K be a field of characteristic 0, and G a group of type E_8 over K. Suppose that $\mathrm{SL}_2(32)$ can be embedded in $G(K)$. Then $\mathrm{PGL}_2(31)$ can be embedded in $G(K)$. Nice!"

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More precisely, Serre proved that $\mathrm{PGL}_2(31)$ can be embedded in $G(K)$ iff G "is compact", i.e., isomorphic to the scalar extension of the anisotropic E_8 over $\mathbb{Q}$, and that $\mathrm{SL}_2(32)$ embeds in $G(K)$ iff G is compact and $\cos(2\pi/11)$ is in K. This led him to the question: How to tell, e.g., if the split E_8 is compact in this sense? We show:

1.1. **Theorem.** $\mathrm{PGL}_2(31)$ *embeds in* $E_8(K)$ *if and only if* -1 *is a sum of* 16 *squares in* K. *The group* $\mathrm{SL}_2(32)$ *embeds in* $E_8(K)$ *if and only if* -1 *is a sum of* 16 *squares in* K *and* $\cos(2\pi/11)$ *is in* K.

2. PRELIMINARIES

Let k denote a field of characteristic 0. We write E_8 for the split simple algebraic group with Killing-Cartan type E_8. The Galois cohomology set $H^1(k, E_8)$ classifies simple algebraic groups of type E_8 over k.

We put

$$H^1(k, E_8)_0 := \{\eta \in H^1(k, E_8) \mid r_{E_8}(\eta) \text{ has odd order}\}.$$

In [Sem 08, Corollary 8.7], the second author defined a morphism of functors:

$$u\colon H^1(*, E_8)_0 \to H^5(*, \mathbb{Z}/2\mathbb{Z}).$$

This is the degree 5 invariant from the title.

Let now G be a group of type E_8. It corresponds with a canonical element of $H^1(k, E_8)$, so it makes sense to speak of "the Rost invariant of G"; we denote it by $r(G)$. Suppose now that $r(G)$ has odd order, so G belongs to $H^1(k, E_8)_0$. The second author also proved in [Sem 08]:

(2.1) $u(G) = 0$ *if and only if there is an odd-degree extension of* k *that splits* G.

For example, the compact group G of type E_8 over $\mathbb{R}$ has Rost invariant zero and $u(G) = (-1)^5$.

As an obvious corollary, we have:

(2.2) *If* k *has cohomological dimension* ≤ 2, *then every* k*-group of type* E_8 *is split by an odd-degree extension of* k.

Serre's "Conjecture II" for groups of type E_8 is that in fact every group of type E_8 over such a field is split.

3. TITS'S CONSTRUCTION OF GROUPS OF TYPE E_8

3.1. There are inclusions of algebraic groups $G_2 \times F_4 \subset E_8$, where G_2 and F_4 denote split groups of those types. Furthermore, this embedding is essentially unique. Applying Galois cohomology gives a function $H^1(k, G_2) \times H^1(k, F_4) \to H^1(k, E_8)$. The first two sets classify octonion k-algebras and Albert k-algebras respectively, so this map gives a construction by Galois descent of groups of type E_8:

$$\boxed{\text{octonion } k\text{-algebras}} \times \boxed{\text{Albert } k\text{-algebras}} \to \boxed{\text{groups of type } E_8}$$

Jacques Tits gave concrete formulas on the level of Lie algebras for this construction in [T], see also [J]. This method of constructing groups of type E_8 is known as the *Tits construction.* (Really, Tits's construction is more general and gives other kinds of groups as well. The variety of possibilities is summarized in Freudenthal's magic square as in [Inv, p. 540]. However, the flavor in all cases is the same, and this case is the most interesting.)

Our purpose is to compute the value of u on those groups of type E_8 with Rost invariant of odd order (so that it makes sense to speak of u) and arising from Tits's construction. We do this in Theorem 3.6.

3.2. Following [Inv], we write $f_3(-)$ for the even component of the Rost invariant of an Albert algebra or an octonion algebra (equivalently, a group of type F_4 or G_2). We write $g_3(-)$ for the odd component of the Rost invariant of an Albert algebra; such algebras also have an invariant f_5 taking values in $H^5(k, \mathbb{Z}/2\mathbb{Z})$. *An Albert algebra A has $g_3(A) = 0$ and $f_5(A) = 0$ iff A has a nonzero nilpotent, iff the group* $\mathrm{Aut}(A)$ *is isotropic.*

Suppose now that $G \in H^1(k, E_8)$ is the image of an octonion algebra O and an Albert algebra A. It follows from a twisting argument as in the proof of Lemma 5.8 in [GQ] — and was pointed out by Rost as early as 1999 — that

$$r(G) = r_{G_2}(O) + r_{F_4}(A).$$

In particular, G belongs to $H^1(k, E_8)_0$ if and only if $f_3(O) + f_3(A) = 0$ in $H^3(k, \mathbb{Z}/2\mathbb{Z})$, i.e., if and only if $f_3(O) = f_3(A)$.

3.3. **Definition.** Define

$$t\colon H^1(*, F_4) \to H^1(*, E_8)_0$$

by sending an Albert k-algebra A to the group of type E_8 constructed from A and the octonion algebra with norm form $f_3(A)$, via Tits's construction from 3.1. By the preceding paragraph, $r(G) = g_3(A) \in H^3(k, \mathbb{Z}/3\mathbb{Z})$, so G does indeed belong to $H^1(k, E_8)_0$.

3.4. **Example.** *If A has a (nonzero) nilpotent element, then the group $t(A)$ is split.* Indeed, $g_3(A)$ is zero so $t(A)$ is in the kernel of the Rost invariant. Also, $t(A)$ is isotropic because it contains the isotropic subgroup $\mathrm{Aut}(A)$, hence $t(A)$ is split by, e.g., [Ga 09b, Prop. 12.1(1)].

3.5. **Example.** In case $k = \mathbb{Q}$ or $\mathbb{R}$, there are exactly three Albert algebras up to isomorphism. All have $g_3 = 0$; they are distinguished by the values of f_3 and f_5.

$f_3(A)$	$f_5(A)$	$t(A)$
0	0	split by Example 3.4
$(-1)^3$	0	split by Example 3.4
$(-1)^3$	$(-1)^5$	anisotropic by [J, p. 118]

It follows from Chernousov's Hasse Principle for groups of type E_8 [PR] that for every number field K with a unique real place, the set $H^1(K, E_8)_0$ has two elements: the split group and the anisotropic group constructed as in the last line of the table.

3.6. **Theorem.** *For every Albert k-algebra A, we have:*

$$u(t(A)) = f_5(A) \quad \in H^5(k, \mathbb{Z}/2\mathbb{Z}).$$

Proof. The composition ut is an invariant $H^1(*, F_4) \to H^5(*, \mathbb{Z}/2\mathbb{Z})$, hence is given by

$$ut(A) = \lambda_5 + \lambda_2 \cdot f_3(A) + \lambda_0 \cdot f_5(A)$$

for uniquely determined elements $\lambda_i \in H^i(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$, see [GMS, p. 50].

We apply this formula to each of the three lines in the table from Example 3.5. Obviously u of the split E_8 is zero, so the first line gives:

$$0 = u(\text{split } E_8) = \lambda_5 \quad \in H^5(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}).$$

Applying this to the second line gives:

$$0 = u(\text{split } E_8) = \lambda_2 \cdot (-1)^3 \in H^5(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}).$$

For the last line, u of the compact E_8 is $(-1)^5$ by (2.1), see the end of [Sem 08] for details. We find:

$$(-1)^5 = u(\text{compact } E_8) = \lambda_0 \cdot (-1)^5,$$

so λ_0 equals 1 in $H^0(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

To show that $\lambda_2 = 0$ we proceed as follows. Consider the purely transcendental extension $F = \mathbb{Q}(x, y, z, a, b)$ and let H be the group of type F_4 with $f_3(H) = (x, y, z)$, $f_5(H) = f_3(H) \cdot (a, b)$ and $g_3(H) = 0$. Then $ut(H) = f_5(H) + f_3(H) \cdot \lambda_2$.

Let K be a generic splitting field for the symbol $f_5(H)$. Since H_K is isotropic, the resulting group $t(H)$ of type E_8 is isotropic over K, and, since it has trivial Rost invariant, it splits over K [Ga 09b, Prop. 12.1]. Obviously, $ut(H)$ is killed by K. Therefore $f_3(H) \cdot \lambda_2$ is zero over K. If $f_3(H) \cdot \lambda_2$ is zero over F, then by taking residues we see that λ_2 is zero in $H^2(\mathbb{Q}(a, b), \mathbb{Z}/2\mathbb{Z})$, hence also in $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$. Otherwise, $f_3(H) \cdot \lambda_2$ is equal to $f_5(H)$ by [OViVo, Theorem 2.1], and again completing and taking residues with respect to the x-, y-, and z-adic valuations, we find that $\lambda_2 = (a, b) \in H^2(\mathbb{Q}(a, b), \mathbb{Z}/2\mathbb{Z})$. But this is impossible because λ_2 is defined over $\mathbb{Q}$. This proves that $\lambda_2 = 0$. □

3.7. *Remark.* In [Sem 08] the second author constructed a ν_4-variety which splits u. Voevodsky has conjectured that if $u \in H^n(k, \mathbb{Z}/2\mathbb{Z})$ and X is a ν_{n-1}-variety which splits u, then u is a pure symbol. Theorem 3.6 confirms this conjecture for the E_8's constructed as in Definition 3.3.

3.8. **Example.** Whatever field k of characteristic zero one starts with, there is an extension K/k that supports an anisotropic 5-Pfister quadratic form q_5—one can adjoin 5 indeterminates to k, for example. Let q_3 be a 3-Pfister form dividing q_5 and let A be the Albert K-algebra with $f_d(A) = e_d(q_d)$ for $d = 3, 5$. The group $G := t(A)$ of type E_8 over K has Rost invariant zero yet $u(G) = f_5(A)$ is nonzero by Theorem 3.6. In particular, G is not split, hence is anisotropic by [Ga 09b, Prop. 12.1(1)].

Example 15.9 in [Ga 09b] produced anisotropic groups of type E_8 in a similar manner, but used the Killing form to see that the resulting groups were anisotropic; that method does not work if -1 is a square in k. Roughly, Example 3.8 above exhibits more anisotropic groups because u is a finer invariant than the Killing form. Specifically, we have:

3.9. **Corollary.** *For every field k of characteristic zero and every group $G \in H^1(k, E_8)_0$ in the image of t, we have:*

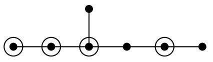
$$\langle 60 \rangle (\mathrm{Kill}_G - \mathrm{Kill}_{E_8}) = 2^3 \cdot u(G) \quad \in I^8(k),$$

where Kill$_-$ *denotes the Killing form of* $-$ *and* E_8 *the split group.*

Proof. Follows from [Ga 09b, 13.5 and Example 15.9] and Theorem 3.6. □

3.10. *Remark.* The invariant u was constructed using motives. One might hope to construct it using methods as in [GMS], in particular to deduce $u(G)$ from the Killing form of G (suggested by analogy with the way that $f_5(A)$ is deduced from the trace form on an Albert algebra A on pages 49–51 of ibid.). But this is impossible. Indeed, using Theorem 3.6 and Corollary 3.9, it is not difficult to construct examples of groups G_1, G_2 of type E_8 such that the Killing forms of G_1 and G_2 are isomorphic but $u(G_1)$ is not equal to $u(G_2)$.

3.11. *Remark* (Application to E_7). Write E_7^{sc} for the split simply connected group of type E_7. The set $H^1(k, E_7^{\mathrm{sc}})_0$ is zero by [Ga 01]. Suppose now that G is simply connected with Tits index



i.e., non-split with a minimal parabolic subgroup that is "wesentlich" in the language of [H, p. 132]. Twisting the inclusion of $(\mathrm{SL}_2 \times E_7)/\mu_2$ in E_8 gives an embedding of G in (the split) E_8 and the composition $H^1(*, G)_0 \to H^1(*, E_8)_0 \xrightarrow{u} H^5(*, \mathbb{Z}/2\mathbb{Z})$ is an invariant. It is not difficult to show that this invariant is not zero if G has Tits algebra $(-1, -1)$ and $(-1)^5 \in H^5(k, \mathbb{Z}/2\mathbb{Z})$ is not zero. (And trivially that $H^1(k, G)_0$ may be nonzero.) One expects that the invariant is nonzero in general and that this follows from Theorem 3.6.

3.12. *Remark.* Recall from [Inv, pp. 436, 437] that the Rost invariant of a class $\eta \in H^1(k, \mathrm{Spin}_{16})$ is given by the formula

$$r_{\mathrm{Spin}_{16}}(\eta) = e_3(q_\eta) \quad \in H^3(k, \mathbb{Z}/2\mathbb{Z})$$

where q_η is the 16-dimensional quadratic form in I^3k corresponding to the image of η in $H^1(k, \mathrm{SO}_{16})$ and e_3 is the Arason invariant. It follows that η belongs to the kernel of the Rost invariant if and only if q_η belongs to I^4k.

Denote by $H^1(*, \mathrm{Spin}_{16})_0$ the kernel of the Rost invariant. One sees using arguments similar to those in [Ga 09a, 18.1, 18.9] that the invariant u of $H^1(*, E_8)_0$ can be used to construct invariants of $H^1(*, \mathrm{Spin}_{16})_0$ of degree 5 and 6, and that the collection of invariants $H^1(*, \mathrm{Spin}_{16})_0 \to H^\bullet(*, \mathbb{Z}/2\mathbb{Z})$ is a rank 5 free $H^\bullet(k, \mathbb{Z}/2\mathbb{Z})$-module with basis consisting of invariants of degree 0, 4, 5, 5, 6. From this it easily follows that for every field of characteristic zero the functor $H^1(*, \mathrm{Spin}_{16})_0$ has essential dimension 6. This should be contrasted with the essential dimension of the functor $H^1(*, \mathrm{Spin}_{16})$, which is 24 by Merkurjev, see [BRV, Remark 3.9].

4. GALOIS DESCENT FOR REPRESENTATIONS OF FINITE GROUPS

In this section, we restate some observations of Serre from [Serre 00] and [GR] regarding projective embeddings of simple groups in exceptional algebraic groups. Combining these results with the u-invariant for E_8 gives some new embeddings results, see Example 4.5 below.

Let A be an abstract finite group and G a semisimple linear algebraic group defined over $\mathbb{Q}$. Fix a faithful representation $\pi\colon G \to \mathrm{GL}_N$ defined over $\mathbb{Q}$.

4.1. **Definition.** Let $\mathbb{Q} \subset F$ be a field. The *character* of a homomorphism $\alpha\colon A \to G(\overline{F})$ is the character of the composition $\pi \circ \alpha\colon A \to \mathrm{GL}_N(\overline{F})$. We say that the character of α is *defined over* F if all its values belong to F.

Let $\varphi\colon A \to G(\overline{F})$ be a monomorphism and χ its character. Assume that χ is defined over F, $Z_{G(\overline{F})}(A) = 1$ (in particular, G is adjoint), that there is exactly one $G(\overline{F})$-conjugacy class of homomorphisms $A \to G(\overline{F})$ with character χ, and G is split or $\mathrm{Aut}\, G = G$.

The following theorem can be extracted from Serre's paper [Serre 00, 2.5.3]:

4.2. **Theorem.** *In the above notation there exists a twisted form G_0 of G defined over F together with a monomorphism $A \to G_0(F)$. Moreover, for a field extension K/F there is a representation $A \to G(K)$ with character χ iff $G \simeq G_0$ over K.*

Proof. Let

$$P = \{\alpha\colon A \to G \mid \alpha \text{ is a representation with character } \chi\};$$

it is a variety over F and G acts on it by conjugation. By assumptions on A and G this action is transitive. Moreover, the condition on the centralizer guarantees that this action is simply transitive, i.e., for any $\alpha, \beta \in P(\overline{F})$ there exists a unique $g \in G(\overline{F})$ with $\beta = \alpha^g$. Thus, P is a G-torsor.

Let $\eta \in H^1(F, G)$ be the 1-cocycle corresponding to the torsor P. Then $\sigma \cdot \varphi = \eta_\sigma^{-1} \varphi \eta_\sigma$ for all σ in the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. Define now G_0 as the twisted form of G over F by the torsor P. The group G_0 is defined out of $G(\overline{F})$ by a twisted Galois action $*$:

$$\sigma * g = \eta_\sigma (\sigma \cdot g) \eta_\sigma^{-1} \qquad (g \in G(\overline{F})).$$

Now it is easy to see that the homomorphism $\varphi\colon A \to G(\overline{F})$ is an F-defined homomorphism $A \to G_0(F)$.

Let K/F be a field extension. If there is a representation $A \to G(K)$ with character χ, then obviously G and G_0 are isomorphic over K. Conversely, if G and G_0 are isomorphic over K, then the image of the cocycle η in $H^1(K, \mathrm{Aut}(G))$ is zero. As G is split adjoint or $\mathrm{Aut}\, G = G$, it follows that η is already zero in $H^1(K, G)$. □

To characterize the isomorphism criterion of Theorem 4.2 we need the following proposition.

4.3. **Proposition.** *For each Killing-Cartan type Φ in the table*

Type Φ	F_4	G_2	E_8
n	3	3	5

there is a unique algebraic group G_0 of type Φ that is compact at every real place of every number field; it is defined over $\mathbb{Q}$. For every field K of characteristic zero and n as in the table, the following are equivalent:

1. *$G_0 \otimes K$ is split.*
2. *$(-1)^n = 0 \in H^n(K, \mathbb{Z}/2)$.*
3. *-1 is a sum of 2^{n-1} squares of the field K.*

Proof. The first sentence is a standard part of the Kneser-Harder-Chernousov Hasse principle. The group G_0 is split at every finite place.

For the second claim, all cases but E_8 are well-known. For E_8, if $G_0 \otimes K$ is split, then $(-1)^5$ is zero by the existence of u; see 2.1. For the converse, G_0 equals $t(A)$ where A is the unique Albert $\mathbb{Q}$-algebra with no nilpotents (see Example 3.5). If $(-1)^5$ — i.e., $f_5(A)$ — is zero in $H^5(K, \mathbb{Z}/2\mathbb{Z})$, then $A \otimes K$ has nilpotents and $G_0 \otimes K$ is split by Example 3.4. □

In the following examples we write Alt_l for the alternating group of degree l and as $\zeta_l = e^{2\pi i/l}$ a primitive l-th root of unity.

4.4. **Example** (type G_2)**.** Let G denote the split group of type G_2, $A = G(\mathbb{F}_2)$ (resp. $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,13)$), and K a field of characteristic zero. Then there is an embedding $A \to G(K)$ iff -1 is a sum of 4 squares of K and $\zeta_9 + \bar{\zeta}_9 \in K$ (for $\mathrm{PSL}(2,8)$), resp. $\sqrt{13} \in K$ (for $\mathrm{PSL}(2,13)$).

Indeed, fix the minimal fundamental representation $G \to \mathrm{GL}_7$. By [A, Theorem 9(3,4,5)] there is a representation $\varphi\colon A \to G(\overline{\mathbb{Q}})$ whose character χ is defined over $F = \mathbb{Q}$ (resp. $F = \mathbb{Q}(\zeta_9 + \bar{\zeta}_9)$, $F = \mathbb{Q}(\sqrt{13})$). Moreover, G acts transitively on the homomorphisms $A \to G(\overline{\mathbb{Q}})$ with character χ (see [A] and [Griess, Cor. 1 and 2]).

By [A, 9.3(1)] the representation φ is irreducible, so $Z_{G(\overline{\mathbb{Q}})}(A) = 1$. Thus, all conditions of Theorem 4.2 are satisfied. Therefore there is a twisted form G_0 of G defined over F and an embedding $A \to G_0(F)$.

In particular, there is an embedding $A \to G_0(\mathbb{R})$. Since any finite subgroup of a Lie group is contained in its maximal compact subgroup, it is easy to see that $G_0 \otimes_F \mathbb{R}$ is compact for all embeddings of F into $\mathbb{R}$. Moreover, by Theorem 4.2 we have an embedding $A \to G(K)$ iff G_0 and G are isomorphic over K. By Proposition 4.3 the latter occurs iff -1 is a sum of 4 squares of K.

(Thus, we have recapitulated the argument from [Serre 00, 2.5.3]).

4.5. **Example** (type E_8)**.** Let G denote the split group of type E_8, $A = \mathrm{PGL}(2,31)$ (resp. $A = \mathrm{SL}(2,32)$), and K a field of characteristic zero. We view G as a subgroup of GL_{248} via the adjoint representation. There is an embedding $A \to G(K)$ iff -1 is a sum of 16 squares (resp. and $\zeta_{11} + \bar{\zeta}_{11} \in K$ for $A = \mathrm{SL}(2,32)$).

Indeed, by [GR, Theorem 2.27 and Theorem 3.25] there exists an embedding $A \to G(\overline{\mathbb{Q}})$ whose character is defined over $F = \mathbb{Q}$ (resp. $F = \mathbb{Q}(\zeta_{11} + \bar{\zeta}_{11})$). Using [GR] one can check all conditions of Theorem 4.2 (cf. Example 4.4).

It follows by Theorem 4.2 that there is an embedding $A \to G_0(F)$ for some twisted form G_0 of G. Again as in Example 4.4 one can see that G_0 is the unique group such that $G_0 \otimes_F \mathbb{R}$ is compact for all embeddings of F into $\mathbb{R}$. Finally by Proposition 4.3 G and G_0 are isomorphic over a field extension K/F iff -1 is a sum of 16 squares in K. This proves Theorem 1.1.

Roughly speaking, we have added the facts about the compact E_8 contained in the proof of Proposition 4.3 (which uses the existence of the u-invariant) to Serre's appendix [GR, App. B].

One can also take G to be the form of E_8 over $\mathbb{Q}$ that is neither split nor anisotropic. Then in the same way one can show that A embeds in $G(K)$ iff -1 is a sum of 4 squares (resp., and $\zeta_{11} + \bar{\zeta}_{11} \in K$ for $A = \mathrm{SL}(2,32)$).

In the same way one can get the following example:

4.6. **Example** (type A_1)**.** Let $G = \mathrm{PGL}_2$, $A = \mathrm{Alt}_4$ (resp. Alt_5), and K a field of characteristic zero. Then there is an embedding $A \to G(K)$ iff -1 is a sum of 2 squares and for Alt_5 additionally $\sqrt{5} \in K$ (see [Serre 72, §2.5] and [Serre 80, §1]).

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