UNRAMIFIED COHOMOLOGY OF CLASSIFYING VARIETIES FOR EXCEPTIONAL SIMPLY CONNECTED GROUPS

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Abstract. Let BG be a classifying variety for an exceptional simple simply connected algebraic group G. We compute the degree 3 unramified Galois cohomology of BG with values in $(\mathbb{Q}/\mathbb{Z})'(2)$ over an arbitrary field F. Combined with a paper by Merkurjev, this completes the computation of these cohomology groups for G semisimple simply connected over all fields.

These computations provide another family of examples of simple simply connected groups G such that BG is not stably rational.

Let G be an algebraic group over a field F with an embedding $\rho\colon G \hookrightarrow SL_n$ over F. We call BG a *classifying space* of G. We will compute the *unramified cohomology* of BG, defined as follows. We write $H^d(F)$ for the Galois cohomology group $H^d(\mathrm{Gal}(F), (\mathbb{Q}/\mathbb{Z})'(d-1))$, where $(\mathbb{Q}/\mathbb{Z})'(d-1) = \varinjlim_n \boldsymbol{\mu}_n^{\otimes(d-1)}$ for n not divisible by the characteristic of F. For each $d \geq 2$, define $H^d_{\mathrm{nr}}(BG/F)$ (or simply $H^d_{\mathrm{nr}}(BG)$) to be the intersection of the kernels of the residue homomorphisms

$$\partial_v\colon H^d(F(BG)) \to H^{d-1}(F(v)) \tag{0.1}$$

as v ranges over the discrete valuations of $F(BG)$ over F. The natural homomorphism $H^d(F) \to H^d_{\mathrm{nr}}(BG/F)$ is split by evaluation at the distinguished point of BG; this gives a direct sum decomposition of $H^d_{\mathrm{nr}}(BG)$, and we denote the complement of $H^d(F)$ by $H^d_{\mathrm{nr}}(BG)_{\mathrm{norm}}$. This group depends only on G and F [M 02, 2.3].

This paper completes the computation of $H^3_{\mathrm{nr}}(BG/F)_{\mathrm{norm}}$ for G semisimple simply connected and F arbitrary. The computation is quickly reduced to the case where G is simple simply connected [M 02, §4]. In [M 02], $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ was computed for G simple and classical. We compute it for the remaining cases, where G is *exceptional*, that is, where G is of type G_2, ${}^3D_4$, ${}^6D_4$, F_4, E_6, E_7, or E_8.

Main Theorem 0.2. *Let G be a simple simply connected* exceptional *algebraic group defined over a field F. Then*

$$H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} = \begin{cases} \mathbb{Z}/2 & \begin{cases} \text{if } \operatorname{char} F \neq 2,\ G \text{ is of type } {}^3D_4, \text{ and } G \text{ has} \\ \text{a nontrivial Tits algebra} \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

(See 5.3 for an explanation of the characteristic $\neq 2$ hypothesis.)

The general motivation for studying $H^d_{\mathrm{nr}}(X)$ is that it can sometimes detect if X is not stably rational, see [C, pp. 35–39]. It was an open question whether BG is stably rational for G semisimple simply connected. The first counterexamples were

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Date: 15 August 2003.

1991 *Mathematics Subject Classification.* 11E76 (17B25 20G10).

provided in [M 02], where Merkurjev exhibited groups G of type ${}^2A_n$, ${}^2D_3$, and ${}^1D_4$ with $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} \neq 0$, hence with BG not stably rational. The results here give another class of such G's, see 7.2.

Our basic tool is that one can compute $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ by inspecting the ramification of the Rost invariant of G, see [M 02] or 5.2. Many questions are settled by hopping along the chain of inclusions

$$G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$$

of split groups, see §6 and 5.5.

The most interesting part of the proof is where we show that the mod 4 portion of the Rost invariant is ramified for groups of type ${}^2E_6$. We prove (in 3.1) that every isotropic trialitarian group embeds in a group of type ${}^2E_6$ with trivial Tits algebras. This settles the question, since the mod 4 portion of the Rost invariant for groups of type ${}^6D_4$ is easily shown to be ramified (6.3). The proof of 3.1 uses Galois descent and interpretations of exceptional groups as acting on nonassociative algebras.

Remark 0.3. Computations of $H^d_{\mathrm{nr}}(X/F)$ in the literature for X a smooth variety (e.g., a classifying variety) typically assume that F is algebraically closed. The examples of nonrational classifying varieties BG provided here and in [M 02] require that F is *not* algebraically closed (e.g., $F = \mathbb{Q}$).

1. VOCABULARY

An (affine) algebraic group is *simple* if it is $\neq 1$, is connected, and has no nontrivial connected normal subgroups over an algebraic closure. (These groups are often called "absolutely almost simple".) Simple groups are classified in, e.g., [KMRT, Ch. VI]. We say that a group is *of type* T_n if it is simple with root system of type T_n and *of type* ${}^tT_n$ if additionally the absolute Galois group of F acts as a group of automorphisms of order t on the Dynkin diagram.

Let V be a finite-dimensional irreducible representation of an algebraic group G over F. The F-algebra $\mathrm{End}_G(V)$ is a skew field by Schur's Lemma, and it is finite-dimensional over F; it is called a *Tits algebra* for G. If it is a (commutative) field, we say that it is *trivial.*

The *Dynkin index* n_G of a simple simply connected algebraic group G is a natural number which depends only upon the type of G and the (Schur) indices of its Tits algebras. The value of n_G can be found in [M 03, App. B], for example:

type of G	G_2	${}^{3,6}D_4$, all Tits alg's trivial	${}^{3,6}D_4$, some Tits alg's nontrivial	F_4	${}^1E_6$	${}^2E_6$	E_7	E_8
n_G	2	6	12	6	6	12	12	60

We have functors $H^1(*, G)$ and $H^3(*)$ which take a field extension of F and give a pointed set and abelian group respectively. A *degree 3 invariant of* G *with values in* $(\mathbb{Q}/\mathbb{Z})'(2)$ is a morphism of functors

$$H^1(*, G) \longrightarrow H^3(*)$$

which takes base points to base points. Such invariants are often called "normalized". We write $\mathrm{Inv}^3(G)$ for the abelian group of such invariants. (Clearly, this definition makes sense for every algebraic group G over F.)

Let $p = \mathrm{char}\, F$ if the characteristic is prime and $p = 1$ otherwise. Write $n_G = p^k n'_G$, where n'_G is a natural number prime to p. The group $\mathrm{Inv}^3(G)$ is cyclic of order n'_G. It has a canonical generator r_G which we call the *Rost invariant* of G. (This is the prime-to-p part of what is called the Rost invariant in [M 03]. I do not know how to define a residue map for the p-primary part.)

For $\alpha\colon H \to G$ a map between simple simply connected algebraic groups over F, there is a positive integer n_α called the *Rost multiplier* or "Dynkin index" of α, see [M 03, §7]. It has the properties: n_H divides $n_\alpha n_G$ and for every extension E of F, the composition

$$H^1(E,H) \xrightarrow{\alpha} H^1(E,G) \xrightarrow{r_G} H^3(E)$$

is $n_\alpha r_H$.

2. $A_2 \subset D_4$

2.1. In this section, we assume that F contains a primitive cube root of unity and hence has characteristic $\neq 3$. Let L be a cubic Galois extension of F; by Kummer theory it is obtained by adjoining a cube root of some element $\lambda \in F^*$. We write (λ) for the corresponding class in $F^*/F^{*3} = H^1(F,\boldsymbol{\mu}_3)$, where $\boldsymbol{\mu}_3$ is the algebraic group of cube roots of unity.

The short exact sequence $1 \to \boldsymbol{\mu}_3 \to SL_3 \to PGL_3 \to 1$ induces a connecting homomorphism $\delta\colon H^1(F,PGL_3) \to H^2(F,\boldsymbol{\mu}_3)$.

Lemma 2.2. *Continue the hypotheses of 2.1. Let G be the quasi-split simply connected group of type ${}^3D_4$ associated with the extension L/F. Then G contains a subgroup isomorphic to PGL_3 such that for every extension E of F the diagram*

$$\begin{tikzcd} H^1(E,PGL_3) \arrow[r, "x\mapsto\delta(x)\cup(\lambda)"] \arrow[d] & H^3(E,\boldsymbol{\mu}_3^{\otimes 2}) \arrow[d] \\ H^1(E,G) \arrow[r, "r_G"] & H^3(E) \end{tikzcd}$$

commutes up to sign, where the arrow on the right comes from the natural map $\boldsymbol{\mu}_3^{\otimes 2} \to (\mathbb{Q}/\mathbb{Z})'(2)$.

Proof. We have maps

$$PGL_3 \times \boldsymbol{\mu}_3 \longrightarrow \mathrm{Spin}_8 \rtimes \mathbb{Z}/3 \longrightarrow F_4 \tag{2.3}$$

where F_4 denotes the split algebraic group of that type and we identify $\boldsymbol{\mu}_3$ with $\mathbb{Z}/3$ using the primitive cube root of unity in F. The first map comes from the fact that $PGL_3 = \mathrm{Aut}(M_3(F))$ preserves the subspace of trace zero elements in $M_3(F)$, see [KMRT, pp. 504, 505]. The second map comes from the Springer decomposition of Albert algebras, see [KMRT, 38.7]. The group $\mathbb{Z}/3$ acts on Spin_8 in a manner which cyclically permutes the vector and half-spin representations and fixes PGL_3 elementwise. The map $\mathrm{Spin}_8 \to F_4$ has Rost multiplier 1.

When we twist the groups in (2.3) by (λ) and restrict to connected components, we obtain the sequence

$$PGL_3 \xrightarrow{\iota} G \xrightarrow{\sigma} F_4 \tag{2.4}$$

The map $H^1(\sigma\iota)$ sends a class $[A] \in H^1(E,PGL_3)$ of a central simple E-algebra of degree 3 to the class of the first Tits construction $[J(A,\lambda)] \in H^1(F,F_4)$, see [KMRT, 39.9]. The composition $r_{F_4} \circ H^1(\sigma\iota)$ is, up to sign, the composition of δ with the cup product $\cdot \cup (\lambda)$ by [KMRT, p. 537]. Since σ has Rost multiplier 1, $r_{F_4} \circ H^1(\sigma) = r_G$, and the lemma is proved. □

3. ${}^3D_4, {}^6D_4 \subset {}^2E_6$

In this section, we assume that F has characteristic $\neq 2$. A simple algebraic group is said to be *trialitarian* if it is of type ${}^3D_4$ or ${}^6D_4$. It is well-known that every quasi-split simply connected trialitarian group is a subgroup of the split F_4, hence of every simply connected quasi-split E_6, and the inclusion has Rost multiplier 1. In this section, we prove:

Theorem 3.1. (char $F \neq 2$) *Let T be a trialitarian simply connected group over F which is F-isotropic but not F-quasi-split. Let K be a quadratic extension of F such that T is K-quasi-split. Then there exists a simply connected group G of type ${}^2E_6$ over F such that*

1. *all of G's Tits algebras are trivial;*
2. *G is of type ${}^1E_6$ over K; and*
3. *T is a subgroup of G with Rost multiplier* 1.

Given a T as in the first sentence of 3.1, such a K always exists by [Ga 98, 0.1]. We postpone the proof of the theorem until the end of this section.

3.2. Let $\mathfrak{C}$ be the split Cayley algebra over F with canonical involution (a.k.a. "conjugation") $\pi_{\mathfrak{C}}$. Fix the basis $u_1, \ldots, u_8$ of $\mathfrak{C}$ as in [Ga 98] and [Ga 01a] so that the bilinear norm form is $\mathfrak{n}(x,y) = x\pi_{\mathfrak{C}}(y) + y\pi_{\mathfrak{C}}(x)$ is given by $\mathfrak{n}(u_i, u_j) = \delta_{(i+j),9}$ (Kronecker delta). Write σ for the involution on $GL(\mathfrak{C})$ which is adjoint for $\mathfrak{n}$.

Let R denote the subgroup of $GL(\mathfrak{C})^{\times 3}$ consisting of so-called *related triples* of proper similitudes of $\mathfrak{n}$, see [Ga 98, §1] or [KMRT, §35] for a definition. This group is reductive with center of rank 2; its derived subgroup consists of triples $\underline{t} = (t_0, t_1, t_2)$ with $t_i \in SO(\mathfrak{n})$ for all i and is isomorphic to Spin_8 [KMRT, 35.7].

The group $S_3 = \langle r, \pi \mid r^3 = \pi^2 = 1, \pi r = r^2 \pi\rangle$ acts on R via

$$ {}^r\underline{t} = (t_1, t_2, t_0) \quad \text{and} \quad {}^\pi(t_0, t_1, t_2) = (\pi_{\mathfrak{C}} t_0 \pi_{\mathfrak{C}}, \pi_{\mathfrak{C}} t_2 \pi_{\mathfrak{C}}, \pi_{\mathfrak{C}} t_1 \pi_{\mathfrak{C}}).$$

Define $R \rtimes S_3$ to be the Cartesian product $R \times S_3$ with multiplication

$$(\underline{t}, \alpha) \cdot (\underline{t}', \beta) = (\underline{t} \cdot {}^\alpha\underline{t}', \alpha\beta).$$

The split Albert algebra J has underlying vector space the matrices in $M_3(\mathfrak{C})$ fixed by the conjugate transpose. With that in mind, we may write a general element of J as in (3.4) below where $\varepsilon_i \in F$, $c_i \in \mathfrak{C}$, and the entries given as $\cdot$ are forced by symmetry. The algebra J has a canonically determined norm form; write $\mathrm{Inv}(J)$ for the group of norm isometries.

There is an injection $g\colon R \rtimes S_3 \hookrightarrow \mathrm{Inv}(J)$ defined by

$$g_{\underline{t}} \begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} := \begin{pmatrix} \mu(t_0)^{-1}\varepsilon_0 & t_2(c_2) & \cdot \\ \cdot & \mu(t_1)^{-1}\varepsilon_1 & t_0(c_0) \\ t_1(c_1) & \cdot & \mu(t_2)^{-1}\varepsilon_2 \end{pmatrix},$$

$$g_r \begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & c_0 & \cdot \\ \cdot & \varepsilon_2 & c_1 \\ c_2 & \cdot & \varepsilon_0 \end{pmatrix}, \text{ and } g_\pi \begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_0 & \pi_{\mathfrak{C}} c_1 & \cdot \\ \cdot & \varepsilon_2 & \pi_{\mathfrak{C}} c_0 \\ \pi_{\mathfrak{C}} c_2 & \cdot & \varepsilon_1 \end{pmatrix}.$$

3.3. Construction of a quasi-split ${}^2E_6$. The algebra J is also endowed with a nondegenerate symmetric bilinear form s defined by

$$s(x,y) = \mathrm{Tr}_J(xy) = \sum_{i=0}^{2} [\varepsilon_i \nu_i + \mathfrak{n}(c_i, d_i)]$$

for

$$x = \begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix}, \qquad y = \begin{pmatrix} \nu_0 & d_2 & \cdot \\ \cdot & \nu_1 & d_0 \\ d_1 & \cdot & \nu_2 \end{pmatrix}. \tag{3.4}$$

For each $f \in GL(J)$, there is a unique $f^\dagger \in GL(J)$ such that $s(f(x), f^\dagger(y)) = s(x, y)$ for all $x, y \in J$.

The map $f \mapsto f^\dagger$ restricts to automorphisms of $\mathrm{Inv}(J)$ and $R \rtimes S_3$ defined over F. We have

$$r^\dagger = r, \quad \pi^\dagger = \pi, \quad \text{and} \quad \underline{t}^\dagger = (\sigma(t_0)^{-1}, \sigma(t_1)^{-1}, \sigma(t_2)^{-1}).$$

Let ι be a generator for $\mathrm{Gal}(K/F)$. We define the groups E_6^K and H to be the groups $\mathrm{Inv}(J)$ and R with twisted ι-actions: For f a K-point, we set ${}^\iota f = \iota f^\dagger \iota$ where the action on the left is the new action and juxtaposition denotes the usual action. The group E_6^K is quasi-split of type ${}^2E_6$.

3.5. *Proof of Theorem 3.1.* Since T is isotropic and not quasi-split, it has a Tits algebra which is a nonsplit quaternion algebra Q over a cubic extension L of F [Ga 98, 0.1]. Put $K = F(\sqrt{b})$. Since T is K-quasi-split, Q is split by $L(\sqrt{b})$, hence Q is of the form $(a, b)_L$ for some $a \in L^*$ such that $N_{L/F}(a) = 1$ [KMRT, 43.9]. Since Q is not split over L, it is not split over the normal closure L^c of L/F [Ga 98, 3.2]. In particular, L^c does not contain a square root of b, so $P = K \otimes_F L^c$ is a quadratic field extension of L^c.

To simplify our argument, we assume that L is not Galois over F, so $\mathrm{Gal}(L^c/F)$ is isomorphic to S_3. (This is the case that will be used in the rest of the paper. The other case — where L is Galois over F — is only easier.) Then, the group $\mathrm{Gal}(P/F)$ is isomorphic to $S_3 \times \boldsymbol{\mu}_2 \cong \mathbb{Z}/6 \rtimes \boldsymbol{\mu}_2$, where the factor of $\mathbb{Z}/6$ corresponds to the subgroup $\mathrm{Gal}(P/\Delta)$ for Δ the unique quadratic extension of F in L^c. We fix generators $\zeta := (r^{-1}, -1) \in S_3 \times \boldsymbol{\mu}_2$ (which generates the copy of $\mathbb{Z}/6$) and $\tau = (\pi, 1)$ (which generates the copy of $\boldsymbol{\mu}_2$ in $\mathbb{Z}/6 \rtimes \boldsymbol{\mu}_2$ corresponding to $\mathrm{Gal}(P/L(\sqrt{b}))$).

We construct the group G by descent as follows. The group E_6^K — and hence $H \rtimes S_3$ — is a closed subgroup of $GL(V)$ for some F-vector space V. (Specifically, E_6^K is the group of algebra automorphisms of a Brown algebra with underlying vector space V, cf. [Ga 01b, 2.9(2)].) We call an additive homomorphism $f \colon V \otimes P \to V \otimes P$ *ϖ-semilinear* if there is some $\varpi \in \mathrm{Gal}(P/F)$ such that

$$f(pv) = \varpi(p) f(v) \quad \text{for all } p \in P \text{ and } v \in V \otimes P.$$

Let $\widetilde{GL}(V)$ denote the (abstract) group of such maps f. We define a group homomorphism $\phi \colon \mathrm{Gal}(P/F) \to \widetilde{GL}(V)$ such that $\phi(\varpi)$ is ϖ-semilinear for all ϖ and $T(F)$ and $G(F)$ are the subgroups of $\mathrm{Spin}_8(P)$ and $E_6^K(P)$ commuting with $\phi(\varpi)$ for all ϖ.

Define $\underline{t} = (t_0, t_1, t_2) \in GL(\mathfrak{C})^{\times 3}$ by setting $t_i = m_i P$, for

$$m_i = \mathrm{diag}\,(1, \rho^i(a), -\rho^i(a), \rho^{i+2}(a)^{-1}, \rho^{i+1}(a)^{-1}, -1, 1, \rho^i(a))$$

with $\rho := \zeta^2$, and P the matrix permuting the basis vectors as $(1\,2)(3\,6)(4\,5)(7\,8)$, for the basis of $\mathfrak{C}$ fixed in 3.2 above. Since $N_{L/F}(a) = 1$, $\underline{t}$ is a related triple by [Ga 98, 1.5(3), 1.6, 1.8]. Set

$$\phi(\zeta) = \underline{t} r \zeta \quad \text{and} \quad \phi(\tau) = \pi\tau. \tag{3.6}$$

We have

$$\zeta \underline{t} \zeta^{-1} = (\sigma(\zeta(t_0)), \sigma(\zeta(t_1)), \sigma(\zeta(t_2)))^{-1} = (\sigma(t_2), \sigma(t_0), \sigma(t_1))^{-1} = r^{-1} \underline{t}^{-1} r,$$

since $\sigma(t_i) = t_i$ for all i. Hence

$$\phi(\zeta)^2 = r^2\zeta^2, \quad \phi(\zeta)^3 = \underline{t}\zeta^3, \quad \text{and}$$

$$\phi(\zeta)^6 = \mathrm{Id}_{V \otimes P}\,. \tag{3.7}$$

Since π and τ commute, we have

$$\phi(\tau)^2 = \mathrm{Id}_{V \otimes P}\,. \tag{3.8}$$

Since

$$\phi(\zeta)^5 = \phi(\zeta)^2 \cdot \phi(\zeta)^3 = \underline{t}r^2\zeta^5,$$

and $\pi\tau$ and $\underline{t}$ commute, it is easy to verify that

$$\phi(\tau)\phi(\zeta) = \phi(\zeta)^5\phi(\tau). \tag{3.9}$$

Equations (3.7), (3.8), and (3.9) give that (3.6) defines a homorphism $\phi\colon \mathrm{Gal}(P/F) \to \widetilde{GL}(V)$. Then the set map $z\colon \mathrm{Gal}(P/F) \to H \rtimes S_3$ defined by $z_\varpi := \phi(\varpi)\varpi^{-1}$ is in fact a 1-cocycle. (This correspondence between groups of semilinear transformations and 1-cocycles is well-explained in [Jac, §3].) Set G to be the twisted group $(E_6^K)_z$; it automatically satisfies (2). Since z takes values in the simply connected group E_6^K, (1) holds.

Since the values of z normalize the subgroup Spin_8 of E_6^K, the twisted group $(\mathrm{Spin}_8)_z$ is a subgroup of G. The inclusion has Rost multiplier 1 since the inclusion $\mathrm{Spin}_8 \subset E_6^K$ over an algebraic closure arises from the natural inclusion of root systems $D_4 \subset E_6$. The restriction of z to Spin_8 is the descent given in [Ga 98, 4.7] to construct T, i.e., $(\mathrm{Spin}_8)_z$ is isomorphic to T, hence (3). $\square$

Remark 3.10. The isotropic group G occurring in Theorem 3.1 is typically not quasi-split, even over L. This can be seen by examining the mod 2 portion of the Rost invariant for $(z) \in H^1(P/L, E_6^K)$, which is typically nontrivial by [Ga 01a, 6.7].

4. A construction

The purpose of this section is to construct a suitable extension of F over which we may apply 3.1:

Proposition 4.1. *Let F be a field of characteristic $\neq 2$, and let K be a quadratic extension of F. There is a regular extension E of F and a group T of type ${}^6D_4$ over E such that T is E-isotropic and $(E \otimes_F K)$-quasi-split, but not E-quasi-split.*

Presumably one could prove 4.1 by applying [M 96] to produce a group T_0 of type ${}^6D_4$ whose Tits algebras have index 8 and then extending scalars to function fields of transfers of generalized Severi-Brauer varieties so that the Tits algebras of T_0 have index 2 and are split by a quadratic extension of the base field. Then by [Ga 98] there is a group T as in 4.1 with the same Tits algebras as in T_0. We give a low-tech argument here.

Lemma 4.2. (char $F \neq 2$) *For $p, q \in F^*$, the ring $L = F(t)[x]/(x^3 + px + qt)$ is a separable cubic field extension of $F(t)$ which is regular over F and not Galois over $F(t)$. There is a prolongation of the t-adic valuation on $F(t)$ to L which is unramified with residue degree 1 and with respect to which x has value 1.*

Proof. If L is not a field, then there is some $a \in F(t)$ such that $a^3 + pa + qt = 0$. Since a is integral over the UFD $F[t]$, it belongs to $F[t]$, so it makes sense to speak of the degree of a. In particular, at least two of the terms a^3, pa, and qt must have the same degree, which is also the maximum of the degrees. This implies that a cannot have positive degree. But then qt, with degree 1, is the unique term of maximal degree, which is a contradiction.

Since p, q are in F^*, the discriminant $-4p^3 - 27q^2t^2$ of L is not 0, hence L is separable over F. An argument similar to the one in the preceding paragraph shows that the discriminant is not a square in $F(t)$: Any square root $b \in F(t)$ of the discriminant would belong to $F[t]$ and have degree 1. Then the coefficient of t in b^2 would be nonzero. Thus L is not Galois over $F(t)$. If $\ell \in L \setminus F(t)$ is algebraic over F, then L is generated by ℓ as an $F(t)$-algebra and the discriminant of the extension $L/F(t)$ comes from F^*/F^{*2}, which is a contradiction. Hence F is algebraically closed in L and L is regular over F.

Hensel's Lemma gives that $x^3 + px + tq$ has a linear factor of the form $x - \pi$ in $F((t))[x]$, where π has t-adic value 1. The map $x \mapsto \pi$ gives an isomorphism of L with the subfield $F(t)(\pi)$ of $F((t))$, and the t-adic valuation obviously extends to L so that x has value 1. Since $F((t))$ is the completion of $F(t)$ with respect to the t-adic valuation and hence is unramified with residue degree 1, the claims about ramification and residue degree of our prolongation to L follow. $\square$

Lemma 4.3. ($\operatorname{char} F \neq 2$) *Let $p, b \in F^*$ be such that the quaternion algebra $(p, b)_F$ is nonsplit. Let L be as in Lemma 4.2. Then the quaternion algebra $(x, b)_L$ is nonsplit and is not isomorphic to $(-qt, b)_L$.*

Proof. Since $N_{L/F(t)}(x) = -qt$, the corestriction of $(x, b)_L$ down to $F(t)$ is Brauer-equivalent to $(-qt, b)_{F(t)}$. This algebra is split if and only if the quadratic form $\langle 1, -b, qt\rangle$ is isotropic over $F(t)$. Over the completion $F((t))$, this form has residue forms $\langle 1, -b\rangle$ and $\langle q\rangle$. Since the algebra $(p, b)_F$ is nonsplit, the first form is anisotropic, hence $\langle 1, -b, qt\rangle$ is anisotropic over $F((t))$ by Springer's Theorem. Thus $(-qt, b)_{F(t)}$ is nonsplit, and hence so is $(x, b)_L$.

For the sake of contradiction, suppose that $(x, b)_L$ is isomorphic to $(-qt, b)_L$, i.e., the algebra $(-xqt, b)_L$ is split. Since

$$-x(qt) = -x(-x^3 - px) = x^4 + px^2 \equiv x^2 + p \mod L^{*2},$$

the algebra $(x^2 + p, b)_L$ is split.

Let $\widehat{L}$ be a completion of L with respect to the prolongation of the t-adic valuation on $F(t)$ given by Lemma 4.2. The norm of $(x^2 + p, b)_L$ is the form $\langle 1, -(x^2 + p), -b, b(x^2 + p)\rangle$ over L. Since x has value 1, over $\widehat{L}$ this form has one residue form $\langle 1, -p, -b, bp\rangle$ over the residue field F. This is the norm of the algebra $(p, b)_F$, which is anisotropic because the algebra is nonsplit. By Springer's Theorem, the norm of $(x^2 + p, b)_L$ is anisotropic over $\widehat{L}$, hence the algebra is not L-split, which contradicts our assumption that $(x, b)_L$ is isomorphic to $(-qt, b)_L$. $\square$

4.4. *Proof of 4.1.* Write K as $F(\sqrt{b})$. Let $F_0 := F(p, t)$ for p, t indeterminates. Set $L_0 := F_0(t)[x]/(x^3 + px + t)$ as in 4.2. Set E to be the function field of the Severi-Brauer variety of the quaternion algebra $(-t, b)_{F_0}$. Since E is regular over F_0, $L := L_0 \otimes_{F_0} E$ is a field which is cubic and not Galois over E; it is the function field of the Severi-Brauer variety of $(-t, b)_{L_0}$.

Since b is not a square in F, the quaternion algebras $(p,b)_{F_0}$ is not split. By 4.3, $(x,b)_{L_0}$ is nonsplit and is not isomorphic to $(-t,b)_{L_0}$, hence $(x,b)_L \cong (x,b)_{L_0} \otimes_{L_0} L$ is not split by a well-known theorem of Amitsur. The corestriction $\operatorname{cor}_{L/E}(x,b)_L$ is $(-t,b)_E$, which is split. Thus there is a simply connected E-isotropic trialitarian group T over E with nontrivial Tits algebra $(x,b)_L$ [Ga 98, 4.7], hence it is not E-quasi-split. It is of type ${}^6D_4$ since L is not Galois over E. It is $E(\sqrt{b})$-quasi-split since (x,b) is split over $L(\sqrt{b})$ [Ga 98, 5.6]. □

5. RAMIFICATION

Let G be an algebraic group over F. We say that an invariant $a \in \operatorname{Inv}^3(G)$ is *unramified* if the composition

$$H^1(E((t)),G) \xrightarrow{a} H^3(E((t))) \xrightarrow{\partial} H^2(E)$$

is 0 for every field extension E of F. Otherwise we say that a is *ramified*. The following example is typical:

Example 5.1. Let F be a field with a primitive cube root of unity, and let $L = F(\lambda^{1/3})$ be a cubic Galois extension of F. We claim that the invariant $a \in \operatorname{Inv}^3(PGL_3)$ from §2 given by $a(x) = \delta(x) \cup (\lambda)$ is ramified.

The set $H^1(F, PGL_3)$ classifies degree 3 cyclic central simple algebras (C,d) for C a cubic Galois extension of F and $d \in F^*$. Let $[C] \in H^1(F, \mathbb{Z}/3)$ denote the class corresponding to C. Then $\delta(C,d) \cup (\lambda)$ is $\pm[C] \cup (d) \cup (\lambda)$ in $H^3(F, \boldsymbol{\mu}_3^{\otimes 2})$. Taking $E = F(u)$ for u an indeterminate and $C = E((t))(t^{1/3})$ a cubic Galois extension of $E((t))$, we have

$$\partial\,[\delta(C,u) \cup (\lambda)] = \pm(u) \cup (\lambda) \quad \in H^2(E).$$

This is nonzero in $H^2(E)$ since u is not a norm from the extension $E(\lambda^{1/3})/E$. Hence a is ramified, as claimed. □

For the rest of the section, we assume that G is simple and simply connected. We write $\operatorname{Inv}^3_{\mathrm{nr}}(G)$ for the subset of unramified invariants in $\operatorname{Inv}^3(G)$. It is a subgroup since ∂ is a group homomorphism.

Lemma 5.2. [M 02] $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}} \cong \operatorname{Inv}^3_{\mathrm{nr}}(G)$. □

In particular, $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ is necessarily finite, see §1.

5.3. If F has prime characteristic p, then multiplication by p is an isomorphism of $(\mathbb{Q}/\mathbb{Z})'(2)$. Hence $H^3(F)$, $\operatorname{Inv}^3(G)$, and — by 5.2 — $H^3_{\mathrm{nr}}(BG/F)_{\mathrm{norm}}$ have no nontrivial p-torsion.

This explains the hypothesis "char $F \neq 2$" in the Main Theorem: a simply connected group G of type ${}^3D_4$ with nontrivial Tits algebras "should" have $H^3_{\mathrm{nr}}(BG)_{\mathrm{norm}}$ equal to $\mathbb{Z}/2$, but this is impossible in characteristic 2.

Strongly Inner Lemma 5.4. *Let G be a simple simply connected group over F, and fix $z \in Z^1(F,G)$. The canonical identification $\operatorname{Inv}^3(G) = \operatorname{Inv}^3(G_z)$ defined by $r_G \leftrightarrow r_{G_z}$ restricts to an identification $\operatorname{Inv}^3_{\mathrm{nr}}(G) = \operatorname{Inv}^3_{\mathrm{nr}}(G_z)$.*

Proof. Since G and G_z are strongly inner forms of each other, they have $n_G = n_{G_z}$, hence $n'_G = n'_{G_z}$.

Let E be an extension of F, and consider the diagram

$$\begin{tikzcd}
H^1(E((t)), G) \arrow[r, "mr_G"] \arrow[d, "\tau_z"', "\cong"] & H^3(E((t))) \arrow[r, "\partial"] \arrow[d, "\cdot - mr_G(z)"'] & H^2(E) \arrow[d, equal] \\
H^1(E((t)), G_z) \arrow[r, "mr_{G_z}"] & H^3(E((t))) \arrow[r, "\partial"] & H^2(E),
\end{tikzcd}$$

where τ_z is the twisting isomorphism and m is an integer. The left box commutes by [Gi 00, p. 76, Lem. 7] or [MPT, 1.7]. The right box commutes because ∂ is a group homomorphism and $\partial(r_G(z)) = 0$. Hence mr_G is ramified if and only if mr_{G_z} is. $\square$

5.5. Functoriality (homomorphisms). Let $\alpha : H \to G$ be a morphism of algebraic groups. Then α induces natural homomorphisms

$$\alpha^* \colon \mathrm{Inv}^3(G) \to \mathrm{Inv}^3(H) \quad \text{and} \quad \alpha^*_{\mathrm{nr}} \colon \mathrm{Inv}^3_{\mathrm{nr}}(G) \to \mathrm{Inv}^3_{\mathrm{nr}}(H).$$

Now suppose that H and G are simple simply connected. If the Rost multiplier of α is 1, then n_H divides n_G, hence n'_G/n'_H divides n_G/n_H. (See §1 for definitions.) Also, α^* is a surjection with kernel of order n'_G/n'_H. Then we have: *If* $\mathrm{Inv}^3_{\mathrm{nr}}(H)$ *is trivial, then* $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ *is* (n_G/n_H)*-torsion.*

5.6. Functoriality (scalar extension). Let K be an extension field of F, and write G_K for $G \times_F K$. The restriction homomorphism

$$\mathrm{res}_{K/F} \colon \mathrm{Inv}^3(G/F) \to \mathrm{Inv}^3(G/K)$$

is the natural surjection $\mathbb{Z}/n'_G \to \mathbb{Z}/n'_{G_K}$; its kernel is the (n'_G/n'_{G_K})-torsion in $\mathrm{Inv}^3(G/F)$. It restricts to a homomorphism

$$(\mathrm{res}_{K/F})_{\mathrm{nr}} \colon \mathrm{Inv}^3_{\mathrm{nr}}(G/F) \to \mathrm{Inv}^3_{\mathrm{nr}}(G/K).$$

The kernel of this map is killed by n'_G/n'_{G_K}, hence by n_G/n_{G_K}. We have: *If* $\mathrm{Inv}^3_{\mathrm{nr}}(G/K)$ *is trivial, then* $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ *is* (n_G/n_{G_K})*-torsion.*

6. The case where G has trivial Tits algebras

In this section, we prove:

Proposition 6.1. *Let G be a simple simply connected exceptional algebraic group defined over a field F. If G has only trivial Tits algebras, then* $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$.

That is, the Main Theorem holds for groups with only trivial Tits algebras by 5.2. In proving the proposition, we may assume that G is quasi-split by the Strongly Inner Lemma 5.4.

The proposition still holds if the hypothesis "exceptional" is dropped; the classical groups are treated in [M 02].

6.2. Type $^3D_4$. Let L be a separable cubic extension of F over which the quasi-split group G is of type $^1D_4$. Since G is quasi-split, $n_{G_L} = 2$ and $\mathrm{Inv}^3_{\mathrm{nr}}(G/L) = 0$ by [M 02, 8.5]. Hence $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ is 3-torsion by 5.6. By 5.3 we may assume that $\mathrm{char}\, F \neq 3$.

Let F' be the extension obtained from F by adjoining (if not already in F) a primitive cube root of unity. Then G is still of type $^3D_4$ over F' and the invariants $2r_G$ and $4r_G$ are ramified over F' by 2.2 and 5.1. Since $2r_G$ and $4r_G$ are the only nontrivial 3-torsion elements of $\mathrm{Inv}^3(G/F)$, we have shown that $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 0.

Lemma 6.3. *Let G be simple simply connected of type ${}^6D_4$ over F (no restriction on the Tits algebras). Then* $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$.

Proof. Let Δ be the unique quadratic extension of F over which G is of type ${}^3D_4$. Let K be a generic quasi-splitting field for G over Δ as in [KR]. Then G is quasi-split of type ${}^3D_4$ over K, hence $n_{G_K} = 6$ and $\mathrm{Inv}^3_{\mathrm{nr}}(G/K) = 0$ by the ${}^3D_4$ case (6.2). We have that $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ is 2-torsion by 5.6.

Let L be a cubic extension of F over which G is of type ${}^2D_4$. Then $n_{G_L} = 2$ or 4 and $\mathrm{Inv}^3_{\mathrm{nr}}(G/L) = 0$ by [M 02, 8.5]. Since $n_G = 6$ or 12 (as $n_{G_L} = 2$ or 4), $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ is 3-torsion.

Combining the two previous paragraphs, we find $\mathrm{Inv}^3_{\mathrm{nr}}(G/F) = 0$. □

6.4. Type G_2. Here $n_G = 2$, so we may assume that $\operatorname{char} F \neq 2$. The Rost invariant for the split group of type G_2 is given explicitly in [KMRT, p. 441]. It is the Elman-Lam invariant for 3-Pfister quadratic forms, which is clearly ramified.

6.5. Type F_4. The split G_2 is contained in our split group G of type F_4 with Rost multiplier 1. Since $n_{F_4} = 6$ and $n_{G_2} = 2$, the group $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 3-torsion by 5.5 and the G_2 case (6.4). In characteristic $\neq 3$, the mod 3 part of the Rost invariant is described in [PR, 3.2], and it is clearly ramified. So $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$.

6.6. Type ${}^1E_6$. The split group G of type E_6 contains a subgroup which is split of type F_4 [Sp, 14.20, 14.24] with Rost multiplier 1 [Ga 01a, 2.4], and we have $n_{F_4} = n_G = 6$. Hence $\mathrm{Inv}^3_{\mathrm{nr}}(G) = 0$ by 5.5 and the F_4 case (6.5).

6.7. Type ${}^2E_6$. The group G is split by a quadratic extension and $n_G = 12$. By 5.6 and the ${}^1E_6$ case (6.6), $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 2-torsion, hence we may assume that $\operatorname{char} F \neq 2$.

Let T and E be as in 4.1. By 3.1, T is contained in a strongly inner form G' of G over E. By Lemma 6.3, $\mathrm{Inv}^3_{\mathrm{nr}}(T/E) = 0$. Since T has a nontrivial Tits algebra over E, the Dynkin index n_T is 12. Since $n_{G'} = 12$ and the inclusion $T \subset G'$ has Rost multiplier 1, we have $\mathrm{Inv}^3_{\mathrm{nr}}(G'/E) = 0$. By the Strongly Inner Lemma, $\mathrm{Inv}^3_{\mathrm{nr}}(G/E) = 0$. Since G is of type ${}^2E_6$ over F and E, we have $n_G = n_{G_E}$, hence $\mathrm{Inv}^3_{\mathrm{nr}}(G/F) = 0$ by 5.6.

6.8. Type E_7. The natural inclusion of root systems gives a split simply connected subgroup of type E_6 inside the split group G of type E_7. Since $n_G = 12$ and $n_{E_6} = 6$, $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 2-torsion by 5.5 and the ${}^1E_6$ case (6.6). Hence we may assume that $\operatorname{char} F \neq 2$.

Set $F' = F(x)$ and $K = F'(\sqrt{x})$. There is a quasi-split simply connected group E_6^K over F of type ${}^2E_6$ associated with the extension K/F'; it injects into $G_{F'}$ with Rost multiplier 1 [Ga 01a, §3]. Since $n_{E_6^K} = n_{G_{F'}} = 12$, $\mathrm{Inv}^3_{\mathrm{nr}}(G/F') = 0$ by 5.5 and the ${}^2E_6$ case (6.7). Since $n_G = 12$, we have $\mathrm{Inv}^3_{\mathrm{nr}}(G/F) = 0$.

6.9. Type E_8. As in the previous cases, we may assume that our group G of type E_8 is actually split. The natural inclusion of root systems gives an embedding of a split simply connected group of type E_7 in G with Rost multiplier 1, so $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 5-torsion by 5.5 and the E_7 case (6.8). In particular, we may assume that $\operatorname{char} F \neq 5$ by 5.3.

Let F' be the extension obtained from F by adjoining two indeterminates and (if necessary) a primitive 5th root of unity. There is an F'-central division algebra D of dimension 5^2, namely the symbol algebra determined by the two indeterminates. Arguing in a manner similar to [Gi 02, §1], one finds a strongly inner form G' of G

over F' and an injection $SL_1(D) \hookrightarrow G'$ with Rost multiplier 1. Now $n_{SL_1(D)} = 5$ [M 03, 11.5] and $\mathrm{Inv}^3_{\mathrm{nr}}(SL_1(D)/F') = 0$ (as can be seen from the explicit formula for the Rost invariant in [M 03, p. 138]), hence $\mathrm{Inv}^3_{\mathrm{nr}}(G'/F')$ is 12-torsion by 5.5. By the Strongly Inner Lemma, $\mathrm{Inv}^3_{\mathrm{nr}}(G/F')$ is 12-torsion. Since the Dynkin index of G is 60 over F and F', $\mathrm{Inv}^3_{\mathrm{nr}}(G/F)$ is 12-torsion by 5.6.

Combining the two preceding paragraphs gives that $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 0. This completes the proof of Prop. 6.1. □

7. Proof of the main theorem

Let G be as in the Main Theorem 0.2. If G has only trivial Tits algebras (e.g., G is of type G_2, F_4, or E_8), then the Main Theorem holds for G by Prop. 6.1 and Lemma 5.2.

If G is of type E_6 or E_7, then we pick a generic quasi-splitting field K of G over F. We have $n_{G_K} = n_G$ and $\mathrm{Inv}^3_{\mathrm{nr}}(G/K) = 0$ (by 6.1), hence $\mathrm{Inv}^3_{\mathrm{nr}}(G/F) = 0$ by 5.6.

If G is of type ${}^6D_4$, the Main Theorem holds by Lemma 6.3. The remaining case is where G is of type ${}^3D_4$ with nontrivial Tits algebras. We have $n_G = 12$ and as in the proof of 6.3, $\mathrm{Inv}^3_{\mathrm{nr}}(G)$ is 2-torsion. Hence we may assume that $\mathrm{char}\, F \neq 2$.

The only nontrivial 2-torsion element of $\mathrm{Inv}^3(G)$ is $6r_G$, so we will complete the proof of the Main Theorem if we show that $6r_G$ is unramified. That is, if we show that for every extension E of F, the composition

$$H^1(E((t)), G) \xrightarrow{6r_{G_{E((t))}}} H^3(E((t))) \xrightarrow{\partial} H^2(E) \tag{7.1}$$

is 0.

If G is of type ${}^1D_4$ over E, then $n_{G_E} = 2$ or 4 [M 03, 15.4]. Hence $6r_{G_{E((t))}} = 2r_{G_{E((t))}}$ and the composition (7.1) is 0 by [M 02, 8.2].

Otherwise, G is of type ${}^3D_4$ over E. That is, if L is a cubic Galois extension of F over which G is of type ${}^1D_4$, the tensor product $L' := L \otimes_F E$ is a cubic field extension of E. We have a diagram

$$\begin{tikzcd}
H^1(E((t)), G) \arrow[r, "6r_{G_{E((t))}}"] \arrow[d, "\mathrm{res}"'] & H^3(E((t))) \arrow[r, "\partial"] \arrow[d, "\mathrm{res}"'] & H^2(E) \arrow[d, "\mathrm{res}_{L'/E}"'] \\
H^1(L'((t)), G) \arrow[r, "6r_{G_{L'((t))}}"] & H^3(L'((t))) \arrow[r, "\partial"] & H^2(L').
\end{tikzcd}$$

The left box commutes because the Rost invariant is compatible with restriction, and the right box commutes because the extension $E((t)) \subset L'((t))$ is unramified, hence the whole diagram commutes. Fix a class α in $H^1(E((t)), G)$. Since $G_{L'((t))}$ is of type ${}^1D_4$, the composition of the two bottom arrows is 0 by the preceding paragraph, and the image of α in $H^2(L')$ is 0. Let $\beta \in H^2(E)$ be the image of α; it is 2-torsion because $6r_G$ is 2-torsion. Hence

$$\beta = 3\beta = \mathrm{cor}_{L'/E}\, \mathrm{res}_{L'/E}(\beta),$$

which is 0 by the commutativity of the diagram. This shows that the composition (7.1) — which is the top row of the diagram — is 0 in this case.

Thus $\mathrm{Inv}^3_{\mathrm{nr}}(G) = \mathbb{Z}/2$ for G of type ${}^3D_4$ with nontrivial Tits algebras when $\mathrm{char}\, F \neq 2$. This completes the proof of the Main Theorem 0.2. □

Corollary 7.2. *Let G be a simply connected group of type ${}^3D_4$ or ${}^6D_4$ over a field F of characteristic $\neq 2$. If G has a nontrivial Tits algebra, then BG is not stably rational.*

Proof. If G is of type ${}^3D_4$, this is a direct consequence of the Main Theorem.

If G is of type ${}^6D_4$, let K be the unique quadratic extension of F over which G is of type ${}^3D_4$. It follows from [Ga 98, 3.2] that G has nontrivial Tits algebras when G is considered as a K-group. Therefore BG is not stably rational as a K-variety, hence not as an F-variety. ☐

Acknowledgements. Thanks to Adrian Wadsworth for his comments on §4 and to Alexander Merkurjev for suggesting the problem and for enlightening conversations.

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