3. Our goal is to write the function in the form \( \frac{1}{1 - r} \), and then use Equation (1) to represent the function as a sum of a power series. \( f(x) = \frac{1}{1 + x} = \frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} (-x)^n \) with \( | -x | < 1 \) \( \iff \) \( |x| < 1 \), so \( R = 1 \) and \( I = (-1, 1) \).

5. \( f(x) = \frac{2}{3} - x = \frac{2}{3} \left( \frac{1}{1 - x/3} \right) = \frac{2}{3} \sum_{n=0}^{\infty} \left( \frac{x}{3} \right)^n \) or, equivalently, \( \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n \). The series converges when \( \left| \frac{x}{3} \right| < 1 \), that is, when \( |x| < 3 \), so \( R = 3 \) and \( I = (-3, 3) \).

7. \( f(x) = \frac{x}{9 + x^2} = \frac{x}{9} \left[ \frac{1}{1 - \left( -\left( \frac{x}{3} \right)^2 \right)} \right] = \frac{x}{9} \sum_{n=0}^{\infty} \left( \frac{x}{3} \right)^{2n} = \frac{x}{9} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{9^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{9^{n+1}} \)

The geometric series \( \sum_{n=0}^{\infty} \left( \frac{x}{3} \right)^2 \) converges when \( \left| \frac{x}{3} \right|^2 < 1 \) \( \iff \) \( \frac{x^2}{9} < 1 \) \( \iff \) \( |x|^2 < 9 \) \( \iff \) \( |x| < 3 \), so \( R = 3 \) and \( I = (-3, 3) \).

9. \( f(x) = \frac{1 + x}{1 - x} = (1 + x) \left( \frac{1}{1 - x} \right) = (1 + x) \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1} = 1 + \sum_{n=1}^{\infty} x^n = 1 + 2 \sum_{n=1}^{\infty} x^n \).

The series converges when \( |x| < 1 \), so \( R = 1 \) and \( I = (-1, 1) \).

A second approach: \( f(x) = \frac{1 + x}{1 - x} = \frac{\frac{1}{1 - x}}{1 - x} = 1 + \frac{2}{1 - x} = -1 + 2 \left( \frac{1}{1 - x} \right) = -1 + 2 \sum_{n=0}^{\infty} x^n = 1 + 2 \sum_{n=1}^{\infty} x^n \).

A third approach:

\( f(x) = \frac{1 + x}{1 - x} = (1 + x) \left( \frac{1}{1 - x} \right) = (1 + x)(1 + x + x^2 + x^3 + \cdots) \)

\( = (1 + x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + x^4 + \cdots) = 1 + 2x + 2x^2 + 2x^3 + \cdots = 1 + 2 \sum_{n=1}^{\infty} x^n. \)

15. \( f(x) = \ln(5 - x) = -\int \frac{dx}{5 - x} = -\frac{1}{5} \int \frac{dx}{1 - x/5} = -\frac{1}{5} \int \left[ \sum_{n=0}^{\infty} \left( \frac{x}{5} \right)^n \right] dx = C - \frac{1}{5} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C - \sum_{n=1}^{\infty} \frac{x^n}{n 5^n} \)

Putting \( x = 0 \), we get \( C = \ln 5 \). The series converges for \( |x/5| < 1 \) \( \iff \) \( |x| < 5 \), so \( R = 5 \).

16. \( f(x) = x^2 \tan^{-1}(x^3) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{2n+1} \) [by Example 7] \( = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3} + 2}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1} \) for \( |x^3| < 1 \) \( \iff \) \( |x| < 1 \), so \( R = 1 \).
11.9 Solutions

17. We know that \( \frac{1}{1 + 4x} = \frac{1}{1 - (-4x)} = \sum_{n=0}^{\infty} (-4x)^n \). Differentiating, we get

\[
\frac{-4}{(1 + 4x)^2} = \sum_{n=1}^{\infty} (-4)^n nx^{n-1} = \sum_{n=0}^{\infty} (-4)^{n+1} (n + 1)x^n,
\]

so

\[
f(x) = \frac{x}{(1 + 4x)^2} = \frac{-x}{4} \cdot \frac{-4}{(1 + 4x)^2} = \frac{-x}{4} \sum_{n=0}^{\infty} (-4)^{n+1} (n + 1)x^n = \sum_{n=0}^{\infty} (-1)^n 4^n (n + 1)x^{n+1}
\]

for \(|-4x| < 1 \iff |x| < \frac{1}{4} \), so \( R = \frac{1}{4} \).

19. By Example 5, \( \frac{1}{(1 - x)^2} = \sum_{n=0}^{\infty} (n + 1)x^n \). Thus,

\[
f(x) = \frac{1 + x}{(1 - x)^2} = \frac{1}{(1 - x)^2} + \frac{x}{(1 - x)^2} = \sum_{n=0}^{\infty} (n + 1)x^n + \sum_{n=0}^{\infty} (n + 1)x^{n+1}
\]

\[
= \sum_{n=0}^{\infty} (n + 1)x^n + \sum_{n=1}^{\infty} nx^n \quad \text{[make the starting values equal]}
\]

\[
= 1 + \sum_{n=1}^{\infty} [(n + 1) + n]x^n = 1 + \sum_{n=1}^{\infty} (2n + 1)x^n = \sum_{n=0}^{\infty} (2n + 1)x^n \text{ with } R = 1.
\]