9.4 Solutions

1. (a) \( \frac{dP}{dt} = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100). \) Comparing to Equation 4,
   \[ \frac{dP}{dt} = kP(1 - P/M), \]
   we see that the carrying capacity is \( M = 100 \) and the value of \( k \) is 0.05.

(b) The slopes close to 0 occur where \( P \) is near 0 or 100. The largest slopes appear to be on the line \( P = 50 \). The solutions are increasing for \( 0 < P_0 < 100 \) and decreasing for \( P_0 > 100 \).

(c) All of the solutions approach \( P = 100 \) as \( t \) increases. As in part (b), the solutions differ since for \( 0 < P_0 < 100 \) they are increasing, and for \( P_0 > 100 \) they are decreasing. Also, some have an IP and some don’t. It appears that the solutions which have \( P_0 = 20 \) and \( P_0 = 40 \) have inflection points at \( P = 50 \).

(d) The equilibrium solutions are \( P = 0 \) (trivial solution) and \( P = 100 \). The increasing solutions move away from \( P = 0 \) and all nonzero solutions approach \( P = 100 \) as \( t \to \infty \).

3. (a) \( \frac{dy}{dt} = ky \left( 1 - \frac{y}{M} \right) \quad \Rightarrow \quad y(t) = \frac{M - y_0}{1 + A e^{-kt}} \quad \text{with} \quad A = \frac{M - y(0)}{y(0)}. \)
   With \( M = 8 \times 10^7, k = 0.71 \), and
   \[ y(0) = 2 \times 10^7, \]
   we get the model \( y(t) = \frac{8 \times 10^7}{1 + 3 e^{-0.71t}} \), so \( y(1) = \frac{8 \times 10^7}{1 + 3 e^{-0.71}} \approx 3.23 \times 10^7 \) kg.

(b) \( y(t) = 4 \times 10^7 \Rightarrow \frac{8 \times 10^7}{1 + 3 e^{-0.71t}} = 4 \times 10^7 \Rightarrow 2 = 1 + 3 e^{-0.71t} \Rightarrow e^{-0.71t} = \frac{1}{3} \Rightarrow -0.71t = \ln \frac{1}{3} \Rightarrow t = \frac{\ln 3}{0.71} \approx 1.55 \) years.

5. Using (7), \( A = \frac{M - P_0}{P_0} = \frac{10,000 - 1000}{1000} = 9 \), so \( P(t) = \frac{10,000}{1 + 9 e^{-kt}} \). \( P(1) = 2500 \Rightarrow 2500 = \frac{10,000}{1 + 9 e^{-k(1)}} \Rightarrow 1 + 9 e^{-k} = 4 \Rightarrow 9 e^{-k} = 3 \Rightarrow e^{-k} = \frac{1}{3} \Rightarrow -k = \ln \frac{1}{3} \Rightarrow k = \ln 3. \) After another three years, \( t = 4 \), and \( P(4) = \frac{10,000}{1 + 9 e^{-\ln 3 t}} = \frac{10,000}{1 + 9 (e^{\ln 3})^{-4}} = \frac{10,000}{1 + 9 (3)^{-4}} = \frac{10,000}{1 + \frac{1}{9}} = \frac{10,000}{\frac{10}{9}} = 9000. \)
7. (a) We will assume that the difference in the birth and death rates is 20 million/year. Let \( t = 0 \) correspond to the year 1990 and use a unit of 1 billion for all calculations. \( k \approx \frac{1}{P} \frac{dP}{dt} = \frac{1}{5.3} (0.02) = \frac{1}{265} \), so

\[
\frac{dP}{dt} = kP \left( 1 - \frac{P}{M} \right) = \frac{1}{265} P \left( 1 - \frac{P}{100} \right), \quad P \text{ in billions}
\]

(b) \( A = \frac{M - P_0}{P_0} = \frac{100 - 5.3}{5.3} = \frac{947}{53} \approx 17.8679 \). \( P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{100}{1 + \frac{947}{53}e^{-t/(1/265)}} \), so \( P(10) \approx 5.49 \) billion.

(c) \( P(110) \approx 7.81 \), and \( P(510) \approx 27.72 \). The predictions are 7.81 billion in the year 2100 and 27.72 billion in 2500.

(d) If \( M = 50 \), then \( P(t) = \frac{50}{1 + \frac{447}{53}e^{-t/(1/265)}} \). So \( P(10) \approx 5.48 \), \( P(110) \approx 7.61 \), and \( P(510) \approx 22.41 \). The predictions become 5.48 billion in the year 2000, 7.61 billion in 2100, and 22.41 billion in the year 2500.

9. (a) Our assumption is that \( \frac{dy}{dt} = ky(1 - y) \), where \( y \) is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (4), \( \frac{dP}{dt} = kP \left( 1 - \frac{P}{M} \right) \), we substitute \( y = \frac{P}{M}, P = My, \) and \( \frac{dP}{dt} = M \frac{dy}{dt} \).

To obtain \( M \frac{dy}{dt} = (M - P_0)(1 - y) \), we substitute \( \frac{dy}{dt} = ky(1 - y) \), our equation in part (a).

Now the solution to (4) is \( P(t) = \frac{M}{1 + Ae^{-kt}} \), where \( A = \frac{M - P_0}{P_0} \).

We use the same substitution to obtain \( My = \frac{M}{1 + \frac{M - M y_0}{M y_0} e^{-kt}} \) \( y = \frac{y_0}{y_0 + (1 - y_0)e^{-kt}} \).

Alternatively, we could use the same steps as outlined in the solution of Equation 4.

(c) Let \( t \) be the number of hours since 8 AM. Then \( y_0 = y(0) = \frac{80}{1000} = 0.08 \) and \( y(1) = \frac{1}{2} \), so

\[
\frac{1}{2} = \frac{0.08}{0.08 + 0.92e^{-4k}} \quad \text{Thus,} \quad 0.08 + 0.92e^{-4k} = 0.16, \quad 0.08 = 0.92 = \frac{2}{23}, \quad \text{and} \quad e^{-k} = \left( \frac{2}{23} \right)^{1/4},
\]

so \( y = \frac{0.08}{0.08 + 0.92(2/23)^{1/4}} = \frac{2}{2 + 23(2/23)^{1/4}} \). Solving this equation for \( t \), we get

\[
2y + 23y \left( \frac{2}{23} \right)^{1/4} = 2 \Rightarrow \left( \frac{2}{23} \right)^{t/4} = \frac{2 - 2y}{23y} \Rightarrow \left( \frac{2}{23} \right)^{t/4} = \frac{2}{23} \cdot \frac{1 - y}{y} \Rightarrow \left( \frac{2}{23} \right)^{t/4 - 1} = \frac{1 - y}{y}.
\]

It follows that \( \frac{t}{4} - 1 = \frac{\ln((1 - y)/y)}{\ln \frac{2}{23}} \), so \( t = 4 \left[ 1 + \frac{\ln((1 - y)/y)}{\ln \frac{2}{23}} \right] \).

When \( y = 0.9 \), \( \frac{1 - y}{y} = \frac{1}{9} \), so \( t = 4 \left[ 1 + \frac{\ln \frac{9}{2}}{\ln \frac{2}{23}} \right] \approx 7.6 \) h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 PM.
11. (a) \( \frac{dP}{dt} = kP \left( 1 - \frac{P}{M} \right) \Rightarrow \frac{d^2P}{dt^2} = k \left[ P \left( -\frac{1}{M} \frac{dP}{dt} \right) + \left( 1 - \frac{P}{M} \right) \frac{dP}{dt} \right] = k \frac{dP}{dt} \left( -\frac{P}{M} + 1 - \frac{P}{M} \right) \\
= k \left[ kP \left( 1 - \frac{P}{M} \right) \right] \left( 1 - \frac{2P}{M} \right) = k^2 P \left( 1 - \frac{P}{M} \right) \left( 1 - \frac{2P}{M} \right) 

(b) \( P \) grows fastest when \( P' \) has a maximum, that is, when \( P'' = 0 \). From part (a), \( P'' = 0 \Leftrightarrow P = 0, P = M, \) or \( P = M/2 \). Since \( 0 < P < M \), we see that \( P'' = 0 \Leftrightarrow P = M/2 \).