1. True. \( y^4 \geq 0, y' = -1 - y^4 < 0 \) and the solutions are decreasing functions.

2. True. 
\[
y = \frac{\ln x}{x} \quad \Rightarrow \quad y' = \frac{1 - \ln x}{x^2}.
\]
LHS = \( x^2 y' + xy = x^2 \cdot \frac{1 - \ln x}{x^2} + x \cdot \frac{\ln x}{x} = (1 - \ln x) + \ln x = 1 \) = RHS, so \( y = \frac{\ln x}{x} \) is a solution of \( x^2 y' + xy = 1 \).

3. False. \( x + y \) cannot be written in the form \( g(x)f(y) \).

4. True. 
\[
y' = 3y - 2x + 6xy - 1 = 6xy - 2x + 3y - 1 = 2x(3y - 1) + 1(3y - 1) = (2x + 1)(3y - 1), \]
so \( y' \) can be written in the form \( g(x)f(y) \), and hence, is separable.

5. True. 
\[
e^x y' = y \quad \Rightarrow \quad y' = e^{-x} y \quad \Rightarrow \quad y' + (e^{-x})y = 0, \]
which is of the form \( y' + P(x)y = Q(x) \), so the equation is linear.

6. False. \( y' + xy = e^y \) cannot be put in the form \( y' + P(x)y = Q(x) \), so it is not linear.

7. True. By comparing \( \frac{dy}{dt} = 2y \left(1 - \frac{y}{5}\right) \) with the logistic differential equation (9.4.4), we see that the carrying capacity is 5; that is, \( \lim_{t \to \infty} y = 5 \).

1. (a) ![Graph](image)
(b) \( \lim_{t \to \infty} y(t) \) appears to be finite for \( 0 \leq c \leq 4 \). In fact 
\[
\lim_{t \to \infty} y(t) = 4 \text{ for } c = 4, \quad \lim_{t \to \infty} y(t) = 2 \text{ for } 0 < c < 4, \text{ and}\n\[
\lim_{t \to \infty} y(t) = 0 \text{ for } c = 0. \text{ The equilibrium solutions are}\n\[
y(t) = 0, y(t) = 2, \text{ and } y(t) = 4.\n\]
2. (a) We sketch the direction field and four solution curves, as shown. Note that the slope \( y' = x/y \) is not defined on the line \( y = 0 \).

(b) \( y' = x/y \iff y \ dy = x \ dx \iff y^2 = x^2 + C \). For \( C = 0 \), this is the pair of lines \( y = \pm x \). For \( C \neq 0 \), it is the hyperbola \( x^2 - y^2 = -C \).

3. (a) We estimate that when \( x = 0.3 \), \( y = 0.8 \), so \( y(0.3) \approx 0.8 \).

(b) \( h = 0.1 \), \( x_0 = 0 \), \( y_0 = 1 \) and \( F(x, y) = x^2 - y^2 \). So \( y_n = y_{n-1} + 0.1(x_{n-1}^2 - y_{n-1}^2) \). Thus,

\[
y_1 = 1 + 0.1(0^2 - 1^2) = 0.9, \quad y_2 = 0.9 + 0.1(0.1^2 - 0.9^2) = 0.82, \quad y_3 = 0.82 + 0.1(0.2^2 - 0.82^2) = 0.75676.
\]

This is close to our graphical estimate of \( y(0.3) \approx 0.8 \).

(c) The centers of horizontal line segments of the direction field are located on the lines \( y = x \) and \( y = -x \). When a solution curve crosses one of these lines, it has a local maximum or minimum.
4. (a) \( h = 0.2, x_0 = 0, y_0 = 1 \) and \( F(x, y) = 2xy^2 \). We need \( y_2 \).

\[
y_1 = 1 + 0.2(2 \cdot 0 \cdot 1^2) = 1, \quad y_2 = 1 + 0.2(2 \cdot 0.2 \cdot 1^2) = 1.08 \approx y(0.4).
\]

(b) \( h = 0.1 \) now, so \( y_1 = 1 + 0.1(2 \cdot 0 \cdot 1^2) = 1, \quad y_2 = 1 + 0.1(2 \cdot 0.1 \cdot 1^2) = 1.02,
\]

\[
y_3 = 1.02 + 0.1(2 \cdot 0.2 \cdot 1.02^2) \approx 1.06162, \quad y_4 = 1.06162 + 0.1(2 \cdot 0.3 \cdot 1.06162^2) \approx 1.1292 \approx y(0.4).
\]

(c) The equation is separable, so we write \( \frac{dy}{y^2} = 2x \ dx \implies \int \frac{dy}{y^2} = \int 2x \ dx \iff -\frac{1}{y} = x^2 + C \), but \( y(0) = 1 \), so

\[
C = -1 \text{ and } y(x) = \frac{1}{1 - x^2} \iff y(0.4) = \frac{1}{1 - 0.16} \approx 1.905. \text{ From this we see that the approximation was greatly improved by increasing the number of steps, but the approximations were still far off.}
\]

5. \( y' = xe^{-\sin x} - y \cos x \implies y' + (\cos x) \ y = xe^{-\sin x} \) (\(*\)). This is a linear equation and the integrating factor is

\[
I(x) = e^{\int \cos x \ dx} = e^{\sin x}. \text{ Multiplying } (*) \text{ by } e^{\sin x} \text{ gives } e^{\sin x} \ y' + e^{\sin x} (\cos x) \ y = x \implies (e^{\sin x} \ y)' = e^{\sin x} \ y = \frac{d}{dx} \left[ e^{\sin x} \ y \right] = \frac{1}{2} x^2 + C \implies y = (\frac{1}{2} x^2 + C) e^{-\sin x}.
\]

6. \( \frac{dx}{dt} = 1 - t + x - tx = (1-t)(1-x) = (1+x)(1-t) \implies \frac{dx}{1+x} = (1-t) \ dt \implies \int \frac{dx}{1+x} = \int (1-t) \ dt \iff \ln|1+x| = t - \frac{1}{2} t^2 + C \iff |1+x| = e^{t-\frac{1}{2}t^2+C} \implies 1+x = \pm e^{t-\frac{1}{2}t^2} \cdot e^C \iff x = -1 + Ke^{t-\frac{1}{2}t^2}, \text{ where } K \text{ is any nonzero constant.}
\]

7. \( 2yc^2 \ y' = 2x + 3 \sqrt{x} \implies 2yc^2 \frac{dy}{dx} = 2x + 3 \sqrt{x} \implies 2yc^2 \ dy = \left(2x + 3 \sqrt{x}\right) dx \implies \int 2yc^2 \ dy = \int \left(2x + 3 \sqrt{x}\right) dx \implies c^2 = x^2 + 2x^{3/2} + C \implies y^2 = \ln(x^2 + 2x^{3/2} + C) \implies y = \pm \sqrt{\ln(x^2 + 2x^{3/2} + C)}
\]

8. \( x^2 y' - y = 2x^2 e^{-1/x} \implies y' - \frac{1}{x^2} \ y = 2xe^{-1/x} \) (\(*\)). This is a linear equation and the integrating factor is

\[
I(x) = e^{\int \left(-\frac{1}{x^2}\right) dx} = e^{1/x}. \text{ Multiplying } (*) \text{ by } e^{1/x} \text{ gives } e^{1/x} \ y' - e^{1/x} \cdot \frac{1}{x^2} \ y = 2x \implies (e^{1/x} \ y)' = 2x \implies e^{1/x} \ y = x^2 + C \implies y = e^{-1/x}(x^2 + C).
\]

9. \( \frac{dr}{dt} + 2tr = r \implies \frac{dr}{dt} = r - 2tr = r(1 - 2t) \implies \int \frac{dr}{r} = \int (1 - 2t) \ dt \implies \ln|r| = t - t^2 + C \implies |r| = e^{t-t^2+C} = ke^{-t^2}. \text{ Since } r(0) = 5, k = 5e^0 = k. \text{ Thus, } r(t) = 5e^{-t^2}.
\]
10. \((1 + \cos x)y' = (1 + e^{-x})\sin x\) \[\Rightarrow \quad \frac{dy}{1 + e^{-x}} = \frac{\sin x \, dx}{1 + \cos x} \quad \Rightarrow \quad \int \frac{dy}{1 + e^{-x}} = \int \frac{\sin x \, dx}{1 + \cos x} \quad \Rightarrow \]

\[\int \frac{e^y \, dy}{1 + e^y} = \int \frac{\sin x \, dx}{1 + \cos x} \quad \Rightarrow \quad \ln |1 + e^y| = \ln |1 + \cos x| + C \quad \Rightarrow \quad \ln(1 + e^y) = -\ln(1 + \cos x) + C \quad \Rightarrow \]

\[1 + e^y = e^{-\ln(1+\cos x)} \cdot e^C \quad \Rightarrow \quad e^y = ke^{-\ln(1+\cos x)} - 1 \quad \Rightarrow \quad y = \ln[ke^{-\ln(1+\cos x)} - 1]. \]

Since \(y(0) = 0\),
\[0 = \ln[ke^{-\ln 2} - 1] \quad \Rightarrow \quad e^0 = k\left(\frac{1}{2}\right) - 1 \quad \Rightarrow \quad k = 4. \]

Thus, \(y(x) = \ln[4e^{-\ln(1+\cos x)} - 1]. \)

An equivalent form is \(y(x) = \ln \frac{3 - \cos x}{1 + \cos x}.\)

11. \(xy' - y = x \ln x \quad \Rightarrow \quad y' - \frac{1}{x} y = \ln x. \quad I(x) = e \int \frac{-1}{x} \, dx = e^{\ln|x|} = \left(e^{\ln|x|}\right)^{-1} = |x|^{-1} = 1/x\) since the condition \(y(1) = 2\) implies that we want a solution with \(x > 0. \)

Multiplying the last differential equation by \(I(x)\) gives

\[\frac{1}{x} y' - \frac{1}{x^2} y = \frac{1}{x} \ln x \quad \Rightarrow \quad \left(\frac{1}{x} y\right)' = \frac{1}{x} \ln x \quad \Rightarrow \quad \frac{1}{x} y = \int \frac{\ln x \, dx}{x} \quad \Rightarrow \quad \frac{1}{x} y = \frac{1}{2}(\ln x)^2 + C \quad \Rightarrow \]

\(y = \frac{1}{2}x(\ln x)^2 + Cx. \)

Now \(y(1) = 2 \quad \Rightarrow \quad 2 = 0 + C \quad \Rightarrow \quad C = 2, \)

so \(y = \frac{1}{2}x(\ln x)^2 + 2x.\)

15. (a) Using (4) and (7) in Section 9.4, we see that for \(\frac{dP}{dt} = 0.1P \left(1 - \frac{P}{2000}\right)\) with \(P(0) = 100, \) we have \(k = 0.1, \)

\(M = 2000, \quad P_0 = 100, \quad A = \frac{2000 - 100}{100} = 19. \)

Thus, the solution of the initial-value problem is

\(P(t) = \frac{2000}{1 + 19e^{-0.1t}} \quad \text{and} \quad P(20) = \frac{2000}{1 + 19e^{-2}} \approx 560. \)

(b) \(P = 1200 \quad \Leftrightarrow \quad 1200 = \frac{2000}{1 + 19e^{-0.1t}} \quad \Leftrightarrow \quad 1 + 19e^{-0.1t} = \frac{2000}{1200} \quad \Leftrightarrow \quad 19e^{-0.1t} = \frac{5}{3} - 1 \quad \Leftrightarrow \)

\(e^{-0.1t} = \left(\frac{2}{3}\right)/19 \quad \Leftrightarrow \quad -0.1t = \ln \left(\frac{2}{3}\right) / 19 \quad \Leftrightarrow \quad t = -10 \ln \left(\frac{2}{3}\right) \approx 33.5. \)
16. (a) Let \( t = 0 \) correspond to 1990 so that \( P(t) = 5.28e^{kt} \) is a starting point for the model. When \( t = 10 \), \( P = 6.07 \).

So \( 6.07 = 5.28e^{10k} \) \( \Rightarrow \) \( 10k = \ln \frac{6.07}{5.28} \) \( \Rightarrow \) \( k = \frac{1}{10} \ln \frac{6.07}{5.28} \approx 0.01394 \). For the year 2020, \( t = 30 \), and

\[ P(30) = 5.28e^{30k} \approx 8.02 \text{ billion.} \]

(b) \( P = 10 \) \( \Rightarrow \) \( 5.28e^{kt} = 10 \) \( \Rightarrow \) \( \frac{10}{5.28} = e^{kt} \) \( \Rightarrow \) \( kt = \ln \frac{10}{5.28} \) \( \Rightarrow \) \( t = \frac{10}{\ln \frac{10}{5.28}} \approx 45.8 \text{ years, that is,} \)

in 1990 + 45 = 2035.

(c) \( P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{100}{1 + Ae^{-kt}} \), where \( A = \frac{100 - 5.28}{5.28} \approx 17.94 \). Using \( k = \frac{1}{10} \ln \frac{6.07}{5.28} \) from part (a), a model is

\[
P(t) \approx \frac{100}{1 + 17.94e^{-0.01394t}} \text{ and } P(30) \approx 7.81 \text{ billion, slightly lower than our estimate of 8.02 billion in part (a).}
\]

(d) \( P = 10 \) \( \Rightarrow \) \( 1 + Ae^{-kt} = \frac{100}{10} \) \( \Rightarrow \) \( Ae^{-kt} = 9 \) \( \Rightarrow \) \( e^{-kt} = 9/A \) \( \Rightarrow \) \( -kt = \ln (9/A) \) \( \Rightarrow \)

\[
t = -\frac{1}{k} \ln \frac{9}{A} \approx 49.47 \text{ years (that is, in 2039), which is later than the prediction of 2035 in part (b).}
\]