MATHEMATICALLY ANNOYING ADVERTISING:

A \cup B = \{x \mid -4 \leq x \leq 4\} \cup \{x \mid -2 \leq x \leq 2\} = \mathbb{R}

When discussing real numbers, it is impossible to get more vague than "up to 15% or more."

FREE!

If someone has paid $x to have the word "FREE" typeset for you and N other people to read, their expected value for the money that will come from you to them is at least $ \frac{x}{N+1}$.

It would be difficult for the phrase "the more you spend the more you save" to be more wrong.
1. True or False. No explanations needed, but no partial credit will be given.

(a) If \( \lim_{n \to \infty} a_n = 0 \), then \( \sum a_n \) is convergent.

False!! The harmonic series \( 1 + 1/2 + 1/3 + 1/4 + \cdots \) is an example where the terms go to zero but the series does not converge.

(b) If \( 0 \leq a_n \leq b_n \) and \( \sum b_n \) converges, then \( \sum a_n \) converges.

True. If the bigger series converges, the smaller one must converge too. This is the Comparison Test.

2. Use either Comparison Test to determine whether the series is convergent or divergent.

\[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \]

By looking at the highest order terms in the numerator and denominator, we know that we want to compare \( a_n = \frac{n}{n^2+1} \) to \( b_n = \frac{n}{n^2} = \frac{1}{n} \).

We have that
\[ \frac{n}{n^2 + 1} \leq \frac{n}{n^2} = \frac{1}{n} \]
and \( \sum \frac{1}{n} \) diverges. This is the wrong direction, so we need to use the Limit Comparison Test.

\[ \lim_{n \to \infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = 1 \]

by using l’Hopital’s rule or dividing the top and bottom by \( n^2 \). So since \( \sum \frac{1}{n} \) diverges, \( \sum \frac{n^2}{n^2+1} \) does too.
3. (a) If we want to apply the Integral Test to the function $f(x)$ defined on $[1, \infty)$ and the series $\sum_{n=1}^{\infty} a_n$, where $a_n = f(n)$, what three properties must $f(x)$ have?

$f(x)$ must be positive, continuous, and decreasing.

Note: Generally, you can see at a glance that a function is positive and continuous. If it’s not obvious that a function is decreasing, you can compute the first derivative and check that it’s eventually negative.

(b) Use the Integral Test to determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

Hint: Use $u$-substitution, and compare your answer to the previous question.

The function $f(x) = \frac{x}{x^2 + 1}$ is positive and continuous for $x \geq 1$. We check that it’s decreasing:

$$f'(x) = \frac{1 \times (x^2 + 1) - x \times (2x)}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2}.$$ 

This is negative for $x > 1$, so we’re good.

Using the $u$-substitution $u = x^2 + 1 \implies du = 2x \, dx$, we get that

$$\int \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 1| + C.$$ 

So the improper integral diverges:

$$\int_{1}^{\infty} \frac{x}{x^2 + 1} \, dx = \lim_{t \to \infty} \frac{1}{2} \ln |x^2 + 1| - \frac{1}{2} \ln |t^2 + 1| = \infty.$$ 

Hence, the series

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

also diverges.