Numbers of the form \( n\sqrt{-1} \) are "imaginary," but can still be used in equations.

\[ \text{Okay.} \]
\[ \text{AND } e^{\pi \sqrt{-1}} = -1. \]
\[ \text{Now you're just messing with me.} \]
1. True or False. You do not need to explain your answer, and no partial credit will be given.

(a) When differentiating or integrating a power series, the interval of convergence stays the same.

False - the radius of convergence stays the same, but the interval could change.

(b) Whenever the Taylor series of \( f(x) \) converges, it is equal to \( f(x) \).

False - The Taylor series does not always equal the function. We did an example of this in class.

2. (a) Find a power series representation for \( \frac{5}{1-4x^2} = 5 \times \frac{1}{1-(4x^2)} \).

\[
5 \times \frac{1}{1-(4x^2)} = 5 \sum_{n=0}^{\infty} (4x^2)^n
\]

(b) Find the interval of convergence for the above power series.

Solution 1: This is a geometric series with ratio \( 4x^2 \), and so it converges when \( |4x^2| < 1 \implies |x^2| < 1/4 \implies |x| < 1/2 \).

Solution 2: Using the ratio test, we have

\[
\frac{|(4x^2)^{n+1}|}{(4x^2)^n} = |4x^2| \to |4x^2| \text{ as } n \to \infty
\]

So the series converges when \( |4x^2| < 1 \implies |x| < 1/2 \) and diverges when \( |4x^2| > 1 \implies |x| > 1/2 \). We need to see what happens when \( x = \pm 1/2 \) to find the interval of convergence. In either case, \( 4x^2 = 1 \), and so the series is just \( 5 \sum 1^n \), which diverges. Therefore, the interval is \( |x| < 1/2 \).
3. (a) Find the Taylor Series for \( f(x) = \sin(x) \) centered at \( a = \frac{\pi}{2} \).

First, we look at the derivatives. Making a table is often helpful for organization:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(\pi/2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sin(x) )</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \cos(x) )</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>(- \sin(x) )</td>
<td>(- 1 )</td>
</tr>
<tr>
<td>3</td>
<td>(- \cos(x) )</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>( \sin(x) )</td>
<td>1</td>
</tr>
</tbody>
</table>

The Taylor series centered at \( a = \frac{\pi}{2} \) is given by

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/2)(x - \pi/a)^n}{n!} = 1 - \frac{(x - \pi/2)^2}{2} + \frac{(x - \pi/2)^4}{4!} - \frac{(x - \pi/2)^6}{6!} + \ldots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n(x - \pi/2)^{2n}}{(2n)!}
\]

(b) Find the radius \( R \) of convergence for the above Taylor series.

We use the ratio test, with

\[
a_n = \frac{(-1)^n(x - \pi/2)^{2n}}{(2n)!}
\]

and

\[
a_{n+1} = \frac{(-1)^{n+1}(x - \pi/2)^{2n+2}}{(2n+2)!}.
\]

Therefore,

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{(x - \pi/2)^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(x - \pi/2)^{2n}} = \left| \frac{(x - \pi/2)^2}{(2n+1)(2n+2)} \right| \to 0 < 1 \quad \text{as } n \to \infty
\]

Therefore, for all \( x \), the Taylor series converges. This means that \( R = \infty \).