The Arc Length Formula: If \( f' \) is continuous on \([a, b]\), then the length of the curve \( y = f(x) \), \( a \leq x \leq b \), is

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx
\]

1. Check that this formula gives us the length we expect for:
   (a) \( y = x \), \( 0 \leq x \leq 1 \)

   \[ f'(x) = 1 \implies [f'(x)]^2 = 1 \]

   \[ L = \int_0^1 \sqrt{2} \, dx = \sqrt{2} \]

   This agrees with the right triangle with side lengths 1, 1, \( \sqrt{2} \).

   (b) \( y = \sqrt{1 - x^2} \), \( 0 \leq x \leq 1 \)

   (This is 1/4th of the circle \( x^2 + y^2 = 1 \))

   \[ f'(x) = \frac{-x}{\sqrt{1 - x^2}} \implies [f'(x)]^2 = \frac{x^2}{1 - x^2} \]

   So

   \[ L = \int_0^1 \sqrt{1 + \frac{x^2}{1 - x^2}} \, dx = \int_0^1 \sqrt{\frac{1}{1-x^2}} \, dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} \]

   We know that a circle with radius 1 has circumference (arc length) \( 2\pi \), and so \( \frac{1}{4} \)th of the circle should have arc length \( \frac{\pi}{2} \).

2. Use the formula to find the arc length of \( y = 1 + \frac{2}{3} x^{3/2} \), \( 0 \leq x \leq 3 \).

   \[ f'(x) = \sqrt{x} \implies [f'(x)]^2 = x \implies L = \int_0^1 \sqrt{1 + x} \, dx = \int_1^2 2u^2 \, du = \frac{2u^3}{3}\bigg|_1^2 = \frac{16}{3} - \frac{2}{3} = \frac{14}{2} \]

   with \( u = \sqrt{1 + x} \implies u^2 - 1 = x \implies 2u \, du = dx \)

3. Set up an integral for the arc length of \( y = \frac{1}{x} \), \( 1 \leq x \leq 2 \).

   Note: It turns out that this integrand does not have an antiderivative made up of functions that we know. So the best we can do it approximate it using one of the methods of 7.7.

   \[ f'(x) = -\frac{1}{x^2} \implies [f'(x)]^2 = \frac{1}{x^4} \implies L = \int_2^1 \sqrt{1 + \frac{1}{x^4}} \, dx \]