CHARACTERIZATIONS OF SYMMETRIC QUASICIRCLES

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1. Introduction

A *quasicircle* in the extended complex plane $\bar{\mathbb{C}}$ is, by the standard definition, the image of the unit circle under a quasiconformal self-mapping of $\bar{\mathbb{C}}$. This notion turns out to be very useful in geometric function theory and other fields of mathematics. A recent survey article of Gehring [Ge] collects over two dozens of characterizations for quasicircles, covering remarkably diverse mathematical contexts in which the quasicircle surfaces as an unexpected but yet perfectly natural object of study. An important subclass of quasicircles, namely symmetric quasicircles, was introduced in the late 1980's by Becker and Pommerenke in the study of conformal mappings [BP, Po]. An interesting fact is that many characterizations for quasicircles have strengthened versions that can be used to characterize symmetric quasicircles. In this article we survey the characterizations of symmetric quasicircles and provide a unified approach to the proofs of several of them.

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2. Core characterizations

In this section we collect several core characterizations that play important roles in deriving other properties of symmetric quasicircles.

2.1. Definitions. Suppose that Ω and Ω' are domains in the extended complex plane $\bar{\mathbb{C}}$ and that $f : \Omega \to \Omega'$ is a homeomorphism. We say that f is a *K-quasiconformal mapping* ($1 \leq K < \infty$) if the local dilatation

$$H_f(z) = \frac{|f_z| + |f_{\bar{z}}|}{||f_z| - |f_{\bar{z}}||}$$

if finite in $\Omega \backslash \{\infty, f^{-1}(\infty)\}$ and $H_f(z) \leq K$ a.e. in Ω, where f_z and $f_{\bar{z}}$ denote partial derivatives. We note that a sense preserving homeomorphism is 1-quasiconformal if and only if it is conformal in the usual sense. A quasiconformal map f is said to be *asympotically conformal* on the unit circle $\mathbb{S}^1$ if it is defined in a neighborhood of $\mathbb{S}^1$ and $H_f(z) \to 1$ as $|z| \to 1$.

Finally, a Jordan curve J is said to be a *symmetric quasicircle* if it is the image of the unit circle under a quasiconformal map of $\bar{\mathbb{C}}$ that is asymptotically conformal on the unit circle. This definition is parallel to the above definition for quasicircles. As we shall see in this paper, it is equivalent to the definitions used in literature ([BP, GS, BY]). To avoid treating the point at infinity separately, we only consider Jordan curves J in the finite plane. But the results here remain true for unbounded Jordan curves as well.

2.2. Strong reverse triangle inequality. A celebrated result of Ahlfors states that a Jordan curve J is a quasicircle if and only if it satisfies the following so-called *reverse triangle inequality*

$$|a - w| + |w - b| \leq c|a - b|$$

for some constant $c \geq 1$ and for all $a, b \in J$ and $w \in J(a, b)$, where $J(a, b)$ denotes the smaller arc of J between a and b. Becker and Pommerenke introduced the concept of strong reverse triangle inequality to define symmetric quasicirlce (or what they called asymptotically conformal curves). A Jordan curve J is said to satisfies the *strong reverse triangle inequality* if

$$\max_{w\in J(a,b)} \frac{|a-w|+|w-b|}{|a-b|} \to 1$$

as $a, b \in J$ and $|a-b| \to 0$. We will show that this geometric property is equivalent to the analytic definition of symmetric quasicircles given above. Becker and Pommerenke showed that this property is equivalent to the following extension property.

2.3. Asymptotically conformal extension. Say a Jordan domain Ω has the asymptotically conformal extension property if each conformal map from the unit disk Δ to Ω has a QC extension to $\mathbb{C}$ that is asymptotically conformal on the unit circle.

2.4. Strong M-condition. Let Ω and Ω^* be the interior domain and exterior domain of a bounded Jordan curve J, respectively. Let $f : \Delta \to \Omega$ and $g : \Delta^* \to \Omega^*$ be conformal maps. Then f and g have homeomorphic extensions to the boundary and one can define homeomorphism $h = g^{-1} \circ f : \mathbb{S}^1 \to \mathbb{S}^1$. Such an h is called a homeomorphism induced by J (or conformal welding). It is well known that J is a quasicircle if and only if the induced homeomorphism h is quasisymmetric in the sense that it satisfies the following M-condition:

$$\frac{|a-w|}{|b-w|} = 1 \Rightarrow \frac{|h(a)-h(w)|}{|h(b)-h(w)|} \le M$$

for some constant $M \ge 1$ and for all $a, w, b \in \mathbb{S}^1$. In order to characterize symmetric quasicircles using the induced homeomorphisms of the unit circle, Gardiner and Sullivan introduced the concept of strong M-condition [GS]. Say a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ satisfies the *strong M-condition* if for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\frac{|a-w|}{|b-w|} = 1 \Rightarrow \frac{|h(a)-h(w)|}{|h(b)-h(w)|} \le 1+\epsilon$$

for all $a, w, b \in \mathbb{S}^1$ with $|a-b| \le \delta$. Such a homeomorphism is labeled *symmetric* by Gardiner and Sullivan. They called a Jordan curve J a symmetric quasicircle if the induced map h is symmetric and showed that this definition is equivalent to Becker-Pommerenke's definition.

2.5. Asymptotically symmetric embeddings. Another interesting characterization of quasicircles states that a Jordan curve J in the complex plane is a quasicircle if and only if it is the image of the unit circle under a quasisymmetric embedding. The idea of quasisymmetric maps was introduced by Ahlfors and Beurling in their study of boundary correspondence of quasiconformal maps of the upper half plane [Ah, BA]. This concept was later promoted by Tukia and Väisälä who introduced and studied quasisymmetric maps between arbitrary metric spaces [TV]. Following their definition, an embedding $f : X \to Y$ (in metric spaces) is called *quasisymmetric*, abbreviated QS, if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\text{(QS)} \qquad \frac{|a - x|}{|b - x|} \leq t \Longrightarrow \frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq \eta(t)$$

for all $a, b, x \in X$ and $t \geq 0$. In this case we also say f is η-QS. Here and in what follows we use the notation $|a - b|$ for the distance between a and b in any metric space. Quasisymmetry turns out to be a very useful tool in the study of analysis and geometry in general metric spaces with no smooth structures (see [HK, He]).

Motivated by a desire to obtain a similar characterization for symmetric quasicircles, Brania and the author recently introduced the following concept [BY]. An embedding $f : X \to Y$ (in metric spaces) is called *asymptotically symmetric*, abbreviated AS, if for any $\epsilon > 0$ and $t > 0$ there is a $\delta > 0$ such that

$$\text{(AS)} \qquad \frac{|a - x|}{|b - x|} \leq t \Longrightarrow \frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq (1 + \epsilon)t$$

for all $a, b, x \in X$ contained in a ball of radius δ. We showed, among other things, that a Jordan curve J in $\mathbb{C}$ is a symmetric quasicircle if and only if it is the image of the unit circle under an asymptotically symmetric embedding $f : S^1 \to \mathbb{C}$.

2.6. Main result. We show here that all the above mentioned characterizations of symmetric quasicircles are equivalent to the definiton given in 2.1.

Theorem 1. *Let J be a Jordan curve in $\mathbb{C}$. Then the following statements are equivalent.*

(A). J is a symmetric quasicircle (in the sense of Definition 2.1);

(B). J satisfies the strong reverse triangle inequality;

(C). J has the asymptotically conformal extension property;

(D). The induced homeomorphism h satisfies the strong M-condition;

(E). J is the image of the unit circle under an AS embedding $f : \mathbb{S}^1 \to \mathbb{C}$.

The original proof of the equivalence of (B), (C), (D) and (E) goes as follows. Becker and Pommerenke first established that (B)⇔(C) using (B) as the definition for a symmetric quasicircle. Then Gardiner and Sullivan showed that (D)⇔(C). Finally, Brania and the author proved that (B)⇔(E). To bring the standard definition (A) into the circle of equivalent definitions, we provide the following unified approach:

$$(A) \Rightarrow (E) \Rightarrow (B) \Rightarrow (C) \Rightarrow (D) \Rightarrow (A).$$

Here we just indicate how one can prove $(A) \Rightarrow (E)$. The remainder is essentially contained in the three articles mentioned above [BP, GS, BY].

2.7. Proof of $(A) \Rightarrow (E)$**.** Let J be a symmetric quasicircle (in the sense of Definition 2.1). Then there is a quasiconformal map f of $\mathbb{C}$ which is asymptotically conformal on the unit circle such that $J = f(\mathbb{S}^1)$. We will show that f is asymptotically symmetric on $\mathbb{S}^1$. Suppose otherwise. Then there exist $\epsilon > 0$ and $t > 0$ such that for each $\delta_n = \frac{1}{n}$ $(n = 1, 2, 3, ...)$ there exist points $a_n, b_n, x_n \in \mathbb{S}^1$ contained in a ball of radius δ_n with the property that

$$|a_n - x_n| \leq t|b_n - x_n|, \quad |f(a_n) - f(x_n)| > (1 + \epsilon)t|f(b_n) - f(x_n)|.$$

By passing to subsequences, we may assume that, as $n \to \infty$, $a_n, b_n, x_n \to x \in \mathbb{S}^1$ for some x, that

$$\frac{|a_n - x_n|}{|b_n - x_n|} \to t_0 \leq t \tag{2.7.1}$$

for some $t_0 \geq 0$ and that

$$\frac{|f(a_n) - f(x_n)|}{|f(b_n) - f(x_n)|} \to (1 + \epsilon_0)t \geq (1 + \epsilon)t \tag{2.7.2}$$

for some $\epsilon_0 > 0$. We may further assume that x_n lies on the smaller component of $\mathbb{S}^1 \backslash \{a_n, b_n\}$. Next, choose appropriate $d_n \in \mathbb{S}^1$ so that

$$\tau_n = \frac{|x_n - b_n||a_n - d_n|}{|a_n - x_n||b_n - d_n|} \to \frac{1}{t_0}, \quad \tau_n' = \frac{|x_n' - b_n'||a_n' - d_n'|}{|a_n' - x_n'||b_n' - d_n'|} \to \frac{1}{(1+\epsilon_0)t},$$

where $p' = f(p)$ for any point $p \in \mathbb{S}^1$. Thus we obtain that

$$\lim_{n\to\infty} \frac{\lambda(\tau_n')}{\lambda(\tau_n)} = \frac{\lambda(1/[(1+\epsilon_0)t])}{\lambda(1/t_0)} > 1, \tag{2.7.3}$$

where $\lambda(\tau)$ is the Teichmüller function that determines the modulus of a Teichmüller ring domain (see [Ah]).

Finally, a contradiction can be obtained by estimating the modulus of the curve family $\Gamma_n' = \Delta(f(E_n), f(F_n); \mathbb{C})$ of curves joining $f(E_n)$ and $f(F_n)$ in $\mathbb{C}$, where E_n and F_n denote the disjoint circular arcs on $\mathbb{S}^1$ from a_n to x_n and from b_n to d_n, respectively. In fact using the assumption that f is asymptotically conformal on the unit circle, routine estimates yield that for any $\epsilon' > 0$ the modulus $M(\Gamma_n')$ satisfies that

$$\lambda(\tau_n') \le M(\Gamma_n') \le (1+\epsilon')\lambda(\tau_n) + \epsilon' \tag{2.7.4}$$

when n is large (see [BY, Proof of Theorem 3.1]). Thus it follows that

$$\lim_{n\to\infty} \frac{\lambda(\tau_n')}{\lambda(\tau_n)} \le 1$$

which contradicts (2.7.3) and hence completes the proof of **(A)** $\Rightarrow$ **(E)**. ■

3. Other characterizations

3.1. Quadrilateral Inequality. We say that Ω has this property if there is a constant $c \ge 1$ such that

$$\frac{1}{c} \le \frac{M(Q)}{M(Q^*)} \le c$$

for all conjugate quadrilaterals Q and Q^* in Ω and Ω^*, respectively. Here $M(Q)$ denotes the modulus of a quadrilateral $Q = \Omega(z_1, z_2, z_3, z_4)$ which consists of a Jordan domain Ω together with four positively oriented vertices z_1, z_2, z_3, z_4 on $\partial\Omega$.

3.2. Extremal Distance Property. We say that Ω has this property if there is a constant $c \geq 2$ such that

$$M(A, B; \mathbb{C}) \leq cM(A, B; \Omega)$$

for any pair of disjoint continua $A, B \subset \partial\Omega$. Here $M(A, B; \Omega)$ denotes the modulus of the curve family joining A and B in a domain Ω.

3.3. Harmonic symmetry property. A Jordan domain Ω is said to have this property if there exist points $z_0 \in \Omega$ and $z_0^* \in \Omega^*$ and a constant $c \geq 1$ such that

$$\omega(z_0^*, \alpha; \Omega^*) \leq c\partial\Omega(z_0^*, \beta; \Omega^*)$$

whenever α, β are two adjacent arcs in $\partial\Omega$ with $\omega(z_0, \alpha; \Omega) = \partial\Omega(z_0, \beta; \Omega)$, where $\omega(z_0, \alpha; \Omega)$ denotes the harmonic measure of an arc with respect to a point in Ω.

It is well known that any one of the above properties can be used to characterize quasicircles. As shown in [WY] and [Ka], similar (but strengthened) properties can also be used to characterize symmetric quasicircles.

3.4. Strong Quadrilateral Inequality. We say that Ω has this property if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{1+\epsilon} \leq \frac{M(Q)}{M(Q^*)} \leq 1 + \epsilon$$

for all conjugate quadrilaterals Q and Q^* in Ω and Ω^* with one of the sides of Q having diameter less than δ.

3.5. Strong Extremal Distance Property. We say that Ω has this property if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$M(A, B; \mathbb{C}) \leq (2 + \epsilon)M(A, B; \Omega)$$

for all continua $A, B \subset \partial\Omega$ with $\min\{\mathrm{diam}(A), \mathrm{diam}(B)\} < \delta$.

3.6. Strong harmonic symmetry property. A Jordan domain Ω is said to have this property if there exist points $z_0 \in \Omega$ and $z_0^* \in \Omega^*$ with the property that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{1+\epsilon} \leq \frac{\omega(z_0^*, \alpha; \Omega^*)}{\omega(z_0^*, \beta; \Omega^*)} \leq 1 + \epsilon$$

whenever α, β are two adjacent arcs in $\partial\Omega$ with

$$\frac{1}{1+\delta} \leq \frac{\omega(z_0, \alpha; \Omega)}{\omega(z_0, \beta; \Omega)} \leq 1+\delta$$

and $\min\{\mathrm{diam}(A), \mathrm{diam}(B)\} < \delta$.

We conclude this note with the following collection of equivalent conditions.

Theorem 2. *Let J be a Jordan curve in $\mathbb{C}$. Then the following statements are equivalent.*

(a). J is a symmetric quasicircle (in the sense of Definition 2.1);

(b). J satisfies the strong quadrilateral inequality;

(c). J has the strong extremal distance property;

(d). J has the strong harmonic symmetry property.

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