Exercises for §1.2

In the next section we will be studying the least upper bound property. Some authors like to use it as the axiom of completeness. In such a treatment the monotone convergence property appears as a theorem. In our treatment, which we chose for its intuitive appeal, it is the least upper bound property that is a theorem!

Exercises for §1.2

1. In Example 1.2.10, let $\lambda = \lim_{n \to \infty} x_n$.
   a. Show that $\lambda$ is a root of $\lambda^2 - \lambda - 2 = 0$.
   b. Find $\lim_{n \to \infty} x_n$.

2. Show that $3^n/n!$ converges to 0.

3. Let $x_n = \sqrt{n^2 + 1} - n$. Compute $\lim_{n \to \infty} x_n$.

4. Let $x_n$ be a monotone increasing sequence such that $x_{n+1} - x_n \leq 1/n$. Must $x_n$ converge?

5. Let $F$ be an ordered field in which every strictly monotone increasing sequence bounded above converges. Prove that $F$ is complete.

§1.3 Least Upper Bounds

The completeness axiom can be put into several other equivalent forms. To state these, we need some further terminology.

1.3.1 Definitions Let $S \subseteq \mathbb{R}$. A number $b$ is called an upper bound for $S$ if for all $x \in S$, we have $x \leq b$.

A number $b$ is called a least upper bound of $S$ if, first, $b$ is an upper bound of $S$ and, second, $b$ is less than or equal to every other upper bound of $S$. See Figure 1.3-1. The least upper bound of $S$ (also called the supremum of $S$) is denoted $\sup S$, sup$(S)$, lub$_S$, or lub$(S)$. If $S \subseteq \mathbb{R}$ is not bounded above (has no upper bound), we say that sup$S$ is infinite and write sup$S = +\infty$.\footnote{If $S$ is empty, then one defines sup$(S)$ to be $\infty$.}
Solution Consider the graph of \( y = x^2 + x \) (Figure 1.3-2). From elementary calculus we see that for \( x = -1/2 \), \( y \) is a minimum. Thus \( S \) may be pictured as shown in Figure 1.3-2.

![Graph of \( y = x^2 + x \)](image)

**FIGURE 1.3-2** The set \( S \) in Example 1.3.5

The \( \sup \) and \( \inf \) clearly occur when \( x^2 + x = 3 \), or, from the quadratic formula, when

\[
x = \frac{-1 \pm \sqrt{1 + 12}}{2} = \frac{-1 \pm \sqrt{13}}{2}.
\]

Thus

\[
\sup S = \frac{\sqrt{13} - 1}{2} \quad \text{and} \quad \inf S = \frac{-(\sqrt{13} + 1)}{2}.
\]

**Exercises for §1.3**

1. Let \( S = \{ x \mid x^3 < 1 \} \). Find \( \sup S \). Is \( S \) bounded below?

2. Consider an increasing sequence \( x_n \) that is bounded above and that converges to \( x \). Let \( S = \{ x_n \mid n = 1, 2, 3, \ldots \} \). Give a plausibility argument that \( x = \sup S \).

3. If \( P \subset Q \subset \mathbb{R} \), \( P \neq \emptyset \), and \( P \) and \( Q \) are bounded above, show that \( \sup P \leq \sup Q \).

4. Let \( A \subset \mathbb{R} \) and \( B \subset \mathbb{R} \) be bounded below and define \( A + B = \{ x+y \mid x \in A \) and \( y \in B \). Is it true that \( \inf(A + B) = \inf A + \inf B \)?

5. Let \( S \subset [0, 1] \) consist of all infinite decimal expansions \( x = 0.a_1a_2a_3 \ldots \) where all but finitely many digits are 5 or 6. Find \( \sup S \).
Exercises for §1.4

Lemma 1.4.6 is reasonable, since a Cauchy sequence has terms that are bunched together—say, by a distance 1 from some point onward—and leaving out a finite number of points does not affect boundedness. As for Lemma 1.4.7, the idea is that the convergent subsequence must drag the entire sequence toward the limit, since all of the terms are close to one another beyond a certain point.

To prove 1.4.4 from 1.4.3, note that by 1.4.6 the Cauchy sequence is bounded, and so by 1.4.3 it has a convergent subsequence. By 1.4.7, the whole Cauchy sequence must converge.

As for Theorem 1.4.3, it is plausible that a bounded sequence has to bunch up somewhere, and from this one can extract a convergent sequence. The main technical difficulty one faces in the precise proof is how to use the completeness of \( \mathbb{R} \); specifically, how to relate the given sequence to a monotone one. Try to think of some ideas before looking at the proof—such attempts, even if unsuccessful, can be of benefit in sharpening your mathematical skills.

1.4.8 Example Let \( x_n \) be a sequence of real numbers with the property that \(|x_n - x_{n+1}| \leq 1/2^n\). Show that \( x_n \) converges.

Solution If we show that \( x_n \) is a Cauchy sequence, the result will follow from Theorem 1.4.4. We can write, by the triangle inequality,

\[
|x_n - x_{n+k}| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{n+k-1} - x_{n+k}|
\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+k-1}} \leq \frac{2}{2^n},
\]

since \(a + ar + ar^2 + \cdots + ar^n = a(1 - r^{n+1})/(1 - r) < a/(1 - r)\) if \(0 < r < 1\). Thus \(|x_n - x_m| \leq 1/2^{n-1}\) if \(m \geq n\). Given \(\varepsilon > 0\), if we choose \(N\) so that \(1/(2^{N-1}) < \varepsilon\), then \(x_n\) satisfies the definition of a Cauchy sequence.

Exercises for §1.4

1. Let \( x_n \) satisfy \(|x_n - x_{n+1}| < 1/3^n\). Show that \( x_n \) converges.
2. Show that the sequence \( x_n = e^{\ln(5n)} \) has a convergent subsequence.
3. Find a bounded sequence with three subsequences converging to three different numbers.
4. Let \( x_n \) be a Cauchy sequence. Suppose that for every \( \varepsilon > 0 \) there is some \( n > 1/\varepsilon \) such that \( |x_n| < \varepsilon \). Prove that \( x_n \to 0 \).

5. True or false: If \( x_n \) is a Cauchy sequence, then for \( n \) and \( m \) large enough, \( d(x_{n+1}, x_{m+1}) \leq d(x_n, x_m) \).

\[ \text{§1.5 Cluster Points; lim inf and lim sup} \]

A useful tool in the study of convergent sequences is the notion of a cluster point:

1.5.1 Definition A point \( x \) is called a cluster point of the sequence \( x_n \) if for every \( \varepsilon > 0 \) there are infinitely many values of \( n \) with \( |x_n - x| < \varepsilon \).

For example, both 1 and \(-1\) are cluster points of the sequence \( 1, -1, 1, -1, 1, -1, \ldots \); notice that this sequence does not converge. However, the next proposition shows that there is a relationship between convergence and cluster points.

1.5.2 Proposition Let \( x_n \) be a sequence in \( \mathbb{R} \) and let \( x \in \mathbb{R} \).

i. \( x \) is a cluster point of \( x_n \) if and only if for each \( \varepsilon > 0 \) and for each \( N \), there is an index \( n > N \) with \( |x_n - x| < \varepsilon \).

ii. \( x \) is a cluster point of \( x_n \) if and only if there is a subsequence of \( x_n \) that converges to \( x \).

iii. \( x_n \to x \) if and only if every subsequence of \( x_n \) converges to \( x \).

iv. \( x_n \to x \) if and only if the sequence is bounded and \( x \) is its only cluster point.

v. \( x_n \to x \) if and only if every subsequence of \( x_n \) has a further subsequence that converges to \( x \).

Consider the sequence \( 1, 0, -1, 1, 0, -1, 1, 0, -1, \ldots \). This sequence also does not converge. It has three cluster points: \( 1, 0, \) and \(-1\). Of these, \( 1 \) and \(-1\)
Exercises for Chapter 1

so that $ab > xy - \varepsilon$, or $|ab - xy| < \varepsilon$. Choose $x$ and $y$ so that $a < x + \varepsilon/(2|b| + 1)$, $b < y + \varepsilon/(2|a|)$, and $b < y + 1$. Then, since $|xy| = |a| |y|$ and $|y| < |b| + 1$, we get

\[
|ab - xy| \leq |ab - ay| + |ay - xy| = |a| |b - y| + |a - x| |y| < |a| \frac{\varepsilon}{2|a|} + \frac{\varepsilon}{2(|b| + 1)}(|b| + 1) = \varepsilon
\]

(using the triangle inequality).

The last assertion can be proven in an analogous way.

\[\varepsilon\]

Exercises for Chapter 1

1. For each of the following sets $S$, find $\sup(S)$ and $\inf(S)$ if they exist:
   a. $\{x \in \mathbb{R} \mid x^2 < 5\}$
   b. $\{x \in \mathbb{R} \mid x^2 > 7\}$
   c. $\{1/n \mid n, \text{ an integer}, n > 0\}$
   d. $\{-1/n \mid n \text{ an integer}, n > 0\}$
   e. $\{3, .33, .333, \ldots\}$

2. Review the proof that $\sqrt{2}$ is irrational. Generalize this to $\sqrt{k}$ for $k$ a positive integer that is not a perfect square.

3. a. Let $x \geq 0$ be a real number such that for any $\varepsilon > 0$, $x \leq \varepsilon$. Show that $x = 0$.
   b. Let $S = [0, 1]$. Show that for each $\varepsilon > 0$ there exists an $x \in S$ such that $x < \varepsilon$.

4. Show that $d = \inf(S)$ iff $d$ is a lower bound for $S$ and for any $\varepsilon > 0$ there is an $x \in S$ such that $d \geq x - \varepsilon$.

5. Let $x_n$ be a monotone increasing sequence bounded above and consider the set $S = \{x_1, x_2, \ldots\}$. Show that $x_n$ converges to $\sup(S)$. Make a similar statement for decreasing sequences.

6. Let $A$ and $B$ be two nonempty sets of real numbers with the property that $x \leq y$ for all $x \in A, y \in B$. Show that there exists a number $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in A, y \in B$. Give a counterexample to this statement for rational numbers (it is, in fact, equivalent to the completeness axiom and is the basis for another way of formulating the completeness axiom known as Dedekind cuts).
46. Prove that each nonempty set $S$ of $\mathbb{R}$ that is bounded above has a least upper bound as follows: Choose $x_0 \in S$ and $M_0$ an upper bound. Let $a_0 = (x_0 + M_0)/2$. If $a_0$ is an upper bound, let $M_1 = a_0$ and $x_1 = x_0$; otherwise let $M_1 = M_0$ and $x_1 > a_0$, $x_1 \in S$. Repeat, generating sequences $x_n$ and $M_n$. Prove that they both converge to $\text{sup}(S)$. 