10. Show that a sequence \( \langle x_n \rangle \) is convergent if and only if there is exactly one extended real number that is a cluster point of the sequence. Is this statement true if we omit the word “extended”?

11. a. Show that a sequence \( \langle x_n \rangle \) which converges to a real number \( l \) is a Cauchy sequence.
   
   b. Show that each Cauchy sequence is bounded.
   
   c. Show that if a Cauchy sequence has a subsequence that converges to \( l \), then the original sequence converges to \( l \).

   d. Establish the Cauchy Criterion: There is a real number \( l \) to which the sequence \( \langle x_n \rangle \) converges if and only if \( \langle x_n \rangle \) is a Cauchy sequence.

12. Show that \( x = \lim x_n \) if and only if every subsequence of \( \langle x_n \rangle \) has in turn a subsequence that converges to \( x \).

13. Show that the real number \( l \) is the limit superior of the sequence \( \langle x_n \rangle \) if and only if (i) given \( \epsilon > 0 \), \( \exists n \) such that \( x_k < l + \epsilon \) for all \( k \geq n \), and (ii) given \( \epsilon > 0 \) and \( n \), \( \exists k \geq n \) such that \( x_k > l - \epsilon \).

14. Show that \( \lim x_n = \infty \) if and only if given \( \Delta \) and \( n \), \( \exists k \geq n \) with \( x_k \geq \Delta \).

   Show that \( \lim x_n \leq \lim x_n \) and that \( \lim x_n = \lim x_n = l \) if and only if \( l = \lim x_n \).

   Prove that

\[
\lim x_n + \lim y_n \leq \lim (x_n + y_n) \leq \lim x_n + \lim y_n,
\]

provided the right and left sides are not of the form \( \infty - \infty \).

17. Prove that if \( x_n > 0 \) and \( y_n \geq 0 \), then

\[
\lim (x_n y_n) \leq (\lim x_n)(\lim y_n),
\]

provided the product on the right is not of the form \( 0 \cdot \infty \).

18. We shall say that a sequence (or series) \( \langle x_n \rangle \) is summable to the real number \( s \) or has a sum \( s \) if the sequence \( \langle s_n \rangle \) defined by \( s_n = \sum_{y=1}^{n} x_y \) has \( s \) as a limit. In this case we write \( s = \sum_{y=1}^{\infty} x_y \). Show that if each \( x_y \geq 0 \), there is always an extended real number \( s \) such that

\[
s = \sum_{y=1}^{\infty} x_y.
\]

19. Show that the series \( \langle x_n \rangle \) has a sum if

\[
\sum_{y=1}^{\infty} |x_y| < \infty.
\]
20. Let \( \langle x_n \rangle \) be a sequence of real numbers. Show that \( x = \lim_{n \to \infty} x_n \) if and only if

\[
x = x_1 + \sum_{r=1}^{\infty} (x_{r+1} - x_r).
\]

21. Let \( E \) be a set of positive real numbers. We define \( \sum_{x \in E} x \) to be \( \sup_{F \in \mathcal{F}} s_F \), where \( \mathcal{F} \) is the collection of finite subsets of \( E \) and \( s_F \) is the (finite) sum of the elements of \( F \).

a. Show that \( \sum_{x \in E} x < \infty \) only if \( E \) is countable.

b. Show that if \( E \) is countable and \( \langle x_n \rangle \) is a one-to-one mapping of \( \mathbb{N} \) onto \( E \), then \( \sum_{x \in E} x = \sum_{n=1}^{\infty} x_n \).

22. Let \( p \) be an integer greater than 1, and \( x \) a real number, \( 0 < x < 1 \). Show that there is a sequence \( \langle a_n \rangle \) of integers with \( 0 \leq a_n < p \) such that

\[
x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}
\]

and that this sequence is unique except when \( x \) is of the form \( \frac{q}{p^n} \), in which case there are exactly two such sequences. Show that, conversely, if \( \langle a_n \rangle \) is any sequence of integers with \( 0 \leq a_n < p \), the series

\[
\sum_{n=1}^{\infty} \frac{a_n}{p^n}
\]

converges to a real number \( x \) with \( 0 \leq x \leq 1 \). If \( p = 10 \), this sequence is called the decimal expansion of \( x \). For \( p = 2 \) it is called the binary expansion; and for \( p = 3 \), the ternary expansion.

23. Show that \( \mathbb{R} \) is uncountable. [Use Problem 1.24. Another proof will be given by Corollary 3.4.]

5 Open and Closed Sets of Real Numbers

The simplest sets of real numbers are the intervals. We define the open interval \((a, b)\) to be the set \( \{x : a < x < b\} \). We always take \( a < b \), but we consider also the infinite intervals \((a, \infty) = \{x : a < x\}\) and \((-\infty, b) = \{x : x < b\}\). Sometimes we write \((-\infty, \infty)\) for the set of all real numbers. We define the closed interval \([a, b]\) to be the set \( \{x : a \leq x \leq b\} \). For closed intervals we take \( a \) and \( b \) finite but always assume that \( a < b \). The half-open interval \((a, b]\) is defined to be
Problems

24. Is the set of rational numbers open or closed?
25. What are the sets of real numbers that are both open and closed?
26. Find two sets $A$ and $B$ such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.
27. Show that $x$ is a point of closure of $E$ if and only if there is a sequence $\langle y_n \rangle$ with $y_n \in E$ and $x = \lim y_n$.
28. A number $x$ is called an accumulation point of a set $E$ if it is a point of closure of $E \sim \{x\}$. Show that the set $E'$ of accumulation points of $E$ is a closed set.
29. Show that $\overline{E} = E \cup E'$.
30. A set is called isolated if $E \cap E' = \emptyset$. Show that every isolated set of real numbers is countable.
31. A set $D$ is called dense in $\mathbb{R}$ if $\overline{D} = \mathbb{R}$. Show that the set of rational numbers is dense in $\mathbb{R}$.
33. Prove Propositions 5 and 7 using Propositions 12, 13, and 14.
34. A point $x$ is called an interior point of a set $A$ if there is a $\delta > 0$ such that the interval $(x - \delta, x + \delta)$ is contained in $A$. The set of interior points of $A$ is denoted by $A^o$. Show that
   a. $A$ is open if and only if $A = A^o$.
   b. $A^o = \sim(\overline{A})$.
35. Derive Proposition 16 from the Heine–Borel Theorem using De Morgan's laws.
36. Let $\langle F_n \rangle$ be a sequence of nonempty closed sets of real numbers with $F_{n+1} \subseteq F_n$. Show that if one of the sets $F_n$ is bounded, then $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$. Give an example to show that this conclusion may be false if we do not require one of the sets to be bounded.

37. The Cantor ternary set $C$ consists of all those real numbers in $[0, 1]$ that have ternary expansion (cf. Problem 22) $\langle a_n \rangle$ for which $a_n$ is never 1. (If $x$ has two ternary expansions, we put $x$ in the Cantor set if one of the expansions has no term equal to 1.) Show that $C$ is a closed set, and that $C$ is obtained by first removing the middle third $(\frac{1}{3}, \frac{2}{3})$ from $[0, 1]$, then removing the middle thirds $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{3}{3})$ of the remaining intervals, and so on.
38. Show that the Cantor set can be put into a one-to-one correspondence with the interval $[0, 1]$.
39. Show that the set of accumulation points of the Cantor set is the Cantor set itself.
whenever \( x < y \). It is called monotone (or strictly monotone) if \( f \) or \(-f\) is monotone (or strictly monotone) increasing. Let \( f \) be a continuous function on the interval \([a, b]\). Then there is a continuous function \( g \) such that \( g(f(x)) = x \) for all \( x \in [a, b] \) if and only if \( f \) is strictly monotone. In this case we also have \( f(g(y)) = y \) for each \( y \) between \( f(a) \) and \( f(b) \). A function \( f \) which has a continuous inverse is called a homeomorphism (between its domain and its range).

47. A continuous function \( \varphi \) on \([a, b]\) is called polygonal (or piecewise linear) if there is a subdivision \( a = x_0 < x_1 < \cdots < x_n = b \) such that \( \varphi \) is linear on each interval \([x_i, x_{i+1}]\). Let \( f \) be an arbitrary continuous function on \([a, b]\) and \( \epsilon \) a positive number. Show that there is a polygonal function \( \varphi \) on \([a, b]\) with \( |f(x) - \varphi(x)| < \epsilon \) for all \( x \in [a, b] \).

48. Let \( x \) be a real number in \([0, 1]\) with the ternary expansion \( \langle a_n \rangle \) (cf. Problem 22). Let \( N = \infty \) if none of the \( a_n \) are 1, and otherwise let \( N \) be the smallest value of \( n \) such that \( a_n = 1 \). Let \( b_n = \frac{1}{2} a_n \) for \( n < N \) and \( b_N = 1 \).

Show that

\[
\sum_{n=1}^{N} \frac{b_n}{2^n}
\]

is independent of the ternary expansion of \( x \) (if \( x \) has two expansions) and that the function \( f \) defined by setting

\[
f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}
\]

is a continuous, monotone function on the interval \([0, 1]\). Show that \( f \) is constant on each interval contained in the complement of the Cantor ternary set (Problem 37), and that \( f \) maps the Cantor ternary set onto the interval \([0, 1]\). (This function is called the Cantor ternary function.)

49. Limit superior of a function of a real variable. Let \( f \) be a real (or extended real) valued function defined for all \( x \) in an interval containing \( y \). We define

\[
\liminf_{x \to y} f(x) = \inf \sup_{\delta>0} \{ f(y) \}
\]

with similar definitions for \( \limsup \).

a. \( \liminf_{x \to y} f(x) \leq A \) if and only if, given \( \epsilon > 0 \), there is a \( \delta > 0 \) such that for all \( x \) with \( 0 < |x - y| < \delta \) we have \( f(x) \leq A + \epsilon \).

b. \( \liminf_{x \to y} f(x) \geq A \) if and only if, given \( \epsilon > 0 \) and \( \delta > 0 \), there is an \( x \) such that \( 0 < |x - y| < \delta \) and \( f(x) \geq A - \epsilon \).
Sec. 6] Continuous Functions

\[ e. \lim_{x \to y} f(x) \leq \lim_{x \to y} f(x) \] with equality (for \( \lim_{x \to y} f \neq \pm \infty \)) if and only if \( \lim_{x \to y} f(x) \) exists.

\[ d. \text{If } \lim_{x \to y} f(x) = A \text{ and } \langle x_n \rangle \text{ is a sequence with } x_n \neq y \text{ such that } y = \lim_{x \to y} x_n \text{, then } \lim_{x \to y} f(x_n) \leq A. \]

\[ e. \text{If } \lim_{x \to y} f(x) = A, \text{ then there is a sequence } \langle x_n \rangle \text{ with } x_n \neq y \text{ such that } y = \lim_{x \to y} x_n \text{ and } A = \lim_{x \to y} f(x_n). \]

\[ f. \text{For a real number } l \text{ we have } l = \lim_{x \to y} f(x) \text{ if and only if } l = \lim_{x \to y} f(x_n) \text{ for every sequence } \langle x_n \rangle \text{ with } x_n \neq y \text{ and } y = \lim_{x \to y} x_n. \]

\[ \text{50. Semicontinuous functions.} \text{ An extended real-valued function } f \text{ is called lower semicontinuous at the point } y \text{ if } f(y) \neq -\infty \text{ and } f(y) \leq \lim_{x \to y} f(x). \]

Similarly, \( f \) is called upper semicontinuous at \( y \) if \( f(y) \neq +\infty \) and \( f(y) \geq \lim_{x \to y} f(x). \) We say that \( f \) is lower (upper) semicontinuous on an interval if it is lower (upper) semicontinuous at each point of the interval. The function \( f \) is upper semicontinuous if and only if the function \( -f \) is lower semicontinuous.

\[ a. \text{Let } f(y) \text{ be finite. Prove that } f \text{ is lower semicontinuous at } y \text{ if and only if given } \varepsilon > 0, \exists \delta > 0 \text{ such that } f(y) \leq f(x) + \varepsilon \text{ for all } x \text{ with } |x - y| < \delta. \]

\[ b. \text{A function } f \text{ is continuous (at a point or in an interval) if and only if it is both upper and lower semicontinuous (at the point or in the interval).} \]

\[ c. \text{Show that a real-valued function } f \text{ is lower semicontinuous on } (a, b) \text{ if and only if the set } \{ x : f(x) > \lambda \} \text{ is open for each real number } \lambda. \]

\[ d. \text{Show that if } f \text{ and } g \text{ are lower semicontinuous functions, so are } f \lor g \text{ and } f + g. \]

\[ e. \text{Let } \langle f_n \rangle \text{ be a sequence of lower semicontinuous functions. Show that the function } f \text{ defined by } f(x) = \sup_n f_n(x) \text{ is also lower semicontinuous.} \]

\[ f. \text{A real-valued function } \varphi \text{ defined on an interval } [a, b] \text{ is called a step function if there is a partition } a = x_0 < x_1 < \cdots < x_n = b \text{ such that for each } i \text{ the function } \varphi \text{ assumes only one value in the interval } (x_i, x_{i+1}). \text{ Show that a step function } \varphi \text{ is lower semicontinuous iff } \varphi(x_i) \text{ is less than or equal to the smaller of the two values assumed in } (x_{i-1}, x_i) \text{ and } (x_i, x_{i+1}). \]

\[ g. \text{A function } f \text{ defined on an interval } [a, b] \text{ is lower semicontinuous if and only if there is a monotone increasing sequence } \langle \varphi_n \rangle \text{ of lower semicontinuous step functions on } [a, b] \text{ such that for each } x \in [a, b] \text{ we have } f(x) = \lim_n \varphi_n(x). \]

\[ h. \text{A function } f \text{ defined on } [a, b] \text{ is lower semicontinuous if and only if is lower semicontinuous.} \]
if there is a monotone increasing sequence $\langle \psi_n \rangle$ of continuous functions such that $f(x) = \lim \psi_n(x)$ for each $x$ in $[a, b]$. [Hint: Modify the functions $\phi_n$ in part (a) to make them continuous.]

i. Prove that a function $f$ that is defined and lower semicontinuous on a closed interval $[a, b]$ is bounded from below and assumes its minimum on $[a, b]$, that is, that there is a $y \in [a, b]$ such that $f(y) \leq f(x)$ for all $x \in [a, b]$.

(ii) Upper and lower envelopes of a function. Let $f$ be a real-valued function defined on $[a, b]$. We define the lower envelope $g$ of $f$ to be the function $g$ defined by

$$g(y) = \sup_{\delta > 0} \inf_{|x-y| < \delta} f(x),$$

and the upper envelope $h$ by

$$h(y) = \inf_{\delta > 0} \sup_{|x-y| < \delta} f(x).$$

a. For each $x \in [a, b]$, $g(x) \leq f(x) \leq h(x)$, and $g(x) = f(x)$ if and only if $f$ is lower semicontinuous at $x$, while $g(x) = h(x)$ if and only if $f$ is continuous at $x$.

b. If $f$ is bounded, the function $g$ is lower semicontinuous, while $h$ is upper semicontinuous.

c. If $\phi$ is any lower semicontinuous function such that $\phi(x) \leq f(x)$ for all $x \in [a, b]$, then $\phi(x) \leq g(x)$ for all $x \in [a, b]$.

7 Borel Sets

Although the intersection of any collection of closed sets is closed and the union of any finite collection of closed sets is closed, the union of a countable collection of closed sets need not be closed. For example, the set of rational numbers is the union of a countable collection of closed sets each of which contains exactly one number. Thus if we are interested in $\sigma$-algebras of sets that contain all of the closed sets, we must consider more general types of sets than the open and closed sets. This leads us to the following definition:

**Definition:** The collection $\mathcal{B}$ of Borel sets is the smallest $\sigma$-algebra which contains all of the open sets.

Such a smallest $\sigma$-algebra exists by Proposition 1.3. It is also the smallest $\sigma$-algebra that contains all closed sets and the smallest $\sigma$-algebra that contains the open intervals.
Sec. 7] Borel Sets

A set which is a countable union of closed sets is called an $F_\sigma$ (F for closed, $\sigma$ for sum). Thus every countable set is an $F_\sigma$, as is, of course, every closed set. A countable union of sets in $F_\sigma$ is again in $F_\sigma$. Since

$$(a, b) = \bigcup_{n=1}^{\infty} \left[ a + \frac{1}{n}, b - \frac{1}{n} \right],$$

each open interval is an $F_\sigma$, and hence each open set is an $F_\sigma$.

We say that a set is a $G_\delta$ if it is the intersection of a countable collection of open sets ($G$ for open, $\delta$ for durchschnitt). Thus the complement of an $F_\sigma$ is a $G_\delta$, and conversely.

The $F_\sigma$ and $G_\delta$ are relatively simple types of Borel sets. We could also consider sets of type $F_{\sigma\delta}$, which are the intersections of countable collections of sets each of which is an $F_\sigma$. Similarly, we can construct the classes $G_{\delta\sigma}$, $F_{\sigma\delta\sigma}$, etc. Thus the classes in the two sequences

$$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, \ldots, G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, \ldots$$

are all classes of Borel sets. However, not every Borel set belongs to one of these classes. Further theory of Borel sets can be found in Kuratowski [11], but we shall need only the properties that follow directly from the fact that they form the smallest $\sigma$-algebra containing the open and closed sets.

Problems

52. Let $f$ be a lower semicontinuous function defined for all real numbers. What can you say about the sets $\{x: f(x) > a\}$, $\{x: f(x) \geq a\}$, $\{x: f(x) < a\}$, $\{x: f(x) \leq a\}$, and $\{x: f(x) = a\}$?

53. Let $f$ be a real-valued function defined for all real numbers. Prove that the set of points at which $f$ is continuous is a $G_\delta$.

54. Let $\langle f_n \rangle$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set $C$ of points where this sequence converges is an $F_{\sigma\delta}$. 

It is also the smallest $\sigma$-algebra.
Since $\epsilon$ was an arbitrary positive number,
\[ m^*(\bigcup A_n) \leq \sum m^* A_n. \]

3. **Corollary:** If $A$ is countable, $m^* A = 0$.

4. **Corollary:** The set $[0, 1]$ is not countable.

5. **Proposition:** Given any set $A$ and any $\epsilon > 0$, there is an open set $O$ such that $A \subset O$ and $m^* O \leq m^* A + \epsilon$. There is a $G \in G_3$ such that $A \subset G$ and $m^* A = m^* G$.

Problems

5. Let $A$ be the set of rational numbers between $0$ and $1$, and let $\{I_n\}$ be a finite collection of open intervals covering $A$. Then $\sum m(I_n) \geq 1$.

6. Prove Proposition 5.

7. Prove that $m^*$ is translation invariant.

8. Prove that if $m^* A = 0$, then $m^*(A \cup B) = m^* B$.

3 Measurable Sets and Lebesgue Measure

While outer measure has the advantage that it is defined for all sets, it is not countably additive. It becomes countably additive, however, if we suitably reduce the family of sets on which it is defined. Perhaps the best way of doing this is to use the following definition due to Carathéodory:

**Definition:** A set $E$ is said to be **measurable** if for each set $A$ we have $m^* A = m^*(A \cap E) + m^*(A \cap \overline{E})$.

Since we always have $m^* A \leq m^*(A \cap E) + m^*(A \cap \overline{E})$, we see that $E$ is measurable if (and only if) for each $A$ we have $m^* A \geq m^*(A \cap E) + m^*(A \cap \overline{E})$. Since the definition of measurability is symmetric in $E$ and $\overline{E}$, we have $E$ measurable whenever $E$ is. Clearly $\emptyset$ and the set $\mathbb{R}$ of all real numbers are measurable.

6. **Lemma:** If $m^* E = 0$, then $E$ is measurable.

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*In the present case $m^*$ is Lebesgue outer measure, and we say that $E$ is *Lebesgue measurable*. More general notions of measurable set are considered in Chapters 11 and 12.*