
(a) $f(x)$ is independent of the choice of the ternary expansion of $x$.

First, one can show that if $x \in [0,1]$ has two expansions, they must have the form

\[
x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} \quad \text{and} \quad x = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n - 1}{3^n} + \sum_{k=n+1}^{\infty} \frac{2}{3^k}.
\]

From this one can deduce that $f(x)$ is independent of the ternary expansion of $x$ by considering two cases: $a_k = 1$ for some $k < n$ or $a_k \neq 1$ for all $k < n$.

(b) Continuity of $f(x)$: Given $x \in [0,1]$ and $\epsilon > 0$, choose $K \geq 1$ such that $2/2^K < \epsilon$. Fix a ternary expansion

\[
x = \sum_{n=1}^{\infty} \frac{a_n(x)}{3^n}
\]

such that there are infinitely many terms with $a_n(x) \neq 2$. Then fix $N_1 \geq K$ such that $a_{N_1}(x) \neq 2$ and let $\delta_1 = 1/3^{N_1}$. Then it follows that

\[
x + \delta_1 = \sum_{n=1}^{N_1-1} \frac{a_n}{3^n} + \frac{a_{N_1} + 1}{3^{N_1}} + \ldots.
\]

Next, fix a ternary expansion of $x$ with infinitely many nonzero coefficients and choose $N_2 \geq K$ such that $a_{N_2}(x) \neq 0$. Let $\delta_1 = 1/3^{N_2}$. Then

\[
x - \delta_2 = \sum_{n=1}^{N_2-1} \frac{a_n}{3^n} + \frac{a_{N_2} - 1}{3^{N_2}} + \ldots.
\]

Based on the above expansions, one can conclude that, when $y \in (x - \delta_2, x + \delta_1)$, $x$ and $y$ have ternary expansions that share the same coefficients for at least the first $K - 1$ terms. Thus, by the definition of $f(x)$, it follows that

\[
|f(x) - f(y)| \leq \sum_{n=K}^{\infty} \frac{1}{2^n} < \epsilon.
\]

(c) Monotonicity of $f(x)$: Let $x < y$ and consider their infinite ternary expansions:

\[
x = \sum_{n=1}^{\infty} \frac{a_n(x)}{3^n}, \quad y = \sum_{n=1}^{\infty} \frac{a_n(y)}{3^n}.
\]
Let $k$ denote the smallest integer such that $a_k(x) < a_k(y)$. If there exists $n < k$ such that $a_n = 1$, then $f(x) = f(y)$ by definition. Otherwise, consider two subcases:

(i) $a_k(x) = 0$. In this case, we have
\[ b_1(x) = b_1(y), \ldots, b_{k-1}(x) = b_{k-1}(y), b_k(x) = 0, b_k(y) = 1, \]
and
\[ f(y) - f(x) = \frac{1}{2k} + \sum_{n=k+1}^{\infty} \frac{b_n(y)}{2^n} - \sum_{n=k+1}^{\infty} \frac{b_n(x)}{2^n} \geq \sum_{n=k+1}^{\infty} \frac{b_n(y)}{2^n} \geq 0. \]

(ii) $a_k(x) = 1$. In this case, we have
\[ b_1(x) = b_1(y), \ldots, b_{k-1}(x) = b_{k-1}(y), b_k(x) = b_k(y) = 1, \]
and
\[ f(x) = \sum_{n=1}^{k} \frac{b_n(x)}{2^n} \leq \sum_{n=1}^{\infty} \frac{b_n(y)}{2^n} = f(y). \]

(d) $f$ is constant on each interval contained in the complement of the Cantor set: Let $I$ be such an interval and let $x, y \in I$ with $x < y$ and
\[ x = \sum_{n=1}^{\infty} \frac{a_n(x)}{3^n}, \quad y = \sum_{n=1}^{\infty} \frac{a_n(y)}{3^n}. \]
As in (e), let $k$ denote the smallest integer such that $a_k(x) < a_k(y)$. We claim that for there exists $n < k$ such that $a_n = 1$. Otherwise, the Cantor number
\[ z = \langle a_1 \ldots a_{k-1}0222 \ldots \rangle \quad \text{(in the case of } a_k(x) = 0) \]
\[ z = \langle a_1 \ldots a_{k-1}2000 \ldots \rangle \quad \text{(in the case of } a_k(x) = 1) \]
belongs to the interval $[x, y]$, which contradicts the assumption. Thus the above claim is true and, by definition, $f(x) = f(y)$ is given by a finite sum.

(e) $f$ maps the Cantor set onto the interval $[0, 1]$: Fix $y \in [0, 1]$ and consider its binary expansion
\[ y = \sum_{n=1}^{\infty} \frac{b_n}{2^n}. \]
Let $a_n = 2b_n$. Then, according to the definition in 2.37, the number
\[ x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \]
is in the Cantor set and we have $f(x) = y$. ■

Problem 2/53: Let $A = \{x : f \text{ is continuous at } x\}$. Fix $x \in A$. Then $f$ is continuous at $x$ $\Rightarrow \forall n = 1, 2, \ldots, \exists \delta_n(x) > 0$ such that
\[ |f(y) - f(x)| < \frac{1}{n} \]
for all $y$ with $|y - x| < \delta_n(x)$.

Let $I_n(x)$ be the open interval $(x - \delta_n(x), x + \delta_n(x))$ and let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{x \in A} I_n(x).$$

Then easy to see that $E$ is a $G_\delta$-set (countable intersection of open set) and that $A \subset E$.

Key: it remains to show that $E \subset A$. Fix $y \in E$ and we want to show that $f$ is continuous at $y$. For any given $\epsilon > 0$, choose $n$ such that $2/n \leq \epsilon$. Then

$$y \in E \Rightarrow y \in \bigcup_{x \in A} I_n(x) \Rightarrow \exists x \in A \text{ such that } y \in I_n(x).$$

Let $\delta = \delta_n(x) - |y - x| > 0$. Then, for $|y' - y| < \delta$, we have

$$|y' - x| = |y' - y| + |y - x| < \delta + |y - x| = \delta_n(x).$$

Thus $y', y \in I_n(x)$ and hence we have

$$|f(y') - f(y)| \leq |f(y') - f(x)| + |f(y) - f(x)| < \frac{2}{n} \leq \epsilon.$$

In summary: $\forall \epsilon > 0 \exists \delta > 0$ such that $|y' - y| < \delta \Rightarrow |f(y') - f(y)| < \epsilon$. This shows that $f$ is continuous at $y \in E$, and hence $A = E$ is a $G_\delta$-set. ■

Another approach to 2/53: Note that $f$ continuous at $x \iff \forall \epsilon > 0, \exists \delta > 0$ such that

$$x_1, x_2 \in (x - \delta, x + \delta) \Rightarrow |f(x_1) - f(x_2)| < \epsilon.$$

Define sets $A_n$ as follows:

$$A_n = \{ x : \exists \delta > 0 \text{ such that } x_1, x_2 \in (x - \delta, x + \delta) \Rightarrow |f(x_1) - f(x_2)| < \frac{1}{n} \}.$$

Easy to see that $A_n$ is open and that $A \subset A_n$ for all $n = 1, 2, ...$. It remains to show that $\bigcap_{n=1}^{\infty} A_n \subset A$. ■

Problem 2/54: Note that for a fixed $x$ the sequence $< f_n >$ converges at $x \iff < f_n(x) >$ is a Cauchy sequence $\iff \forall k = 1, 2, ..., \exists N \geq 1$ such that

$$n, m \geq N \Rightarrow |f_n(x) - f_m(x)| \leq \frac{1}{k}.$$

Thus the set $C$ of convergence can be written as

$$C = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n, m \geq N} \{ x : |f_n(x) - f_m(x)| \leq \frac{1}{k} \}.$$

Then one can verify that $C$ is an $F_{\sigma\delta}$-set. ■
Problem 3/5: We observe that

\[ [0, 1] = \bar{A} \subset \bigcup I_n = \bigcup \bar{I}_n \text{ (since it is a finite union)}. \]

Thus it follows that

\[ 1 = m^*[0, 1) \leq m^* (\bigcup \bar{I}_n) \leq \sum m^*(\bar{I}_n) = \sum |I_n|. \]

Another approach: We can label the covering intervals \( I_i = (a_i, b_i) \) so that \( a_1 < a_2 < ... < a_n \) and \( b_1 < b_2 < ... < b_n \). Then it is easy to see that \( a_1 \leq 0 \) and \( b_n \geq 1 \). Furthermore, since \( A \) is dense in the interval \( [0, 1] \), we have \( a_{i+1} \leq b_i \) (otherwise, the interval \((b_i, a_{i+1})\) is not covered). Thus

\[ \sum_{i=1}^{n} l(I_i) = (b_1 - a_1) + (b_2 - a_2) + ... + (b_n - a_n) \geq (a_2 - a_1) + (a_3 - a_2) + ... + (a_n - a_{n-1}) + (b_n - a_n) \geq b_n - a_1 \geq 1 - 0 = 1. \]

Problem 3/6: (i). Given set \( A \) and \( \epsilon > 0 \), by the definition of outer measure, there is an open cover \( \{I_n\} \) of \( A \) such that \( \sum l(I_n) \leq m^*(A) + \epsilon \). Let \( O = \bigcup I_n \). Then \( O \) is open with \( A \subset O \) and, by subadditivity,

\[ m^*(O) \leq \sum_{n} m^*(I_n) = \sum_{n} l(I_n) \leq m^*(A) + \epsilon. \]

(ii). By (i), for any \( n \) there is an open set \( O_n \) with \( A \subset O_n \) and \( m^*(O_n) \leq m^*(A) + 1/n \). Let \( G = \cap_n O_n \). Then \( G \) is a \( G_\delta \)–set and \( A \subset G \subset O_n \). Thus

\[ m^*(A) \leq m^*(G) \leq m^*(O_n) \leq m^*(A) + \frac{1}{n} \]

for all \( n \geq 1 \). This shows that \( m^*(G) = m^*(A) \) as desired. \( \blacksquare \)