A note on rational representations in Schmüdgen's Theorem

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Introduction

For a field F, $F[X]$ denotes $F[x_1, \ldots, x_n]$, and for a commutative ring A, $\sum A^2$ denotes the set of sums of squares of elements in A. Given $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[X]$, K_S denotes denotes the basic closed semialgebraic set generated by S, i.e., $K_S = \{\alpha \in \mathbb{R}^n \mid g_i(\alpha) \geq 0 \text{ for } i = 1, \ldots, s\}$. T_S denotes the preordering in $\mathbb{R}[X]$ generated by S, i.e., T_S consists of elements of the form

$$\sum_{\epsilon \in \{0,1\}^n} \sigma_\epsilon g_1^{\epsilon_1} \cdots g_s^{\epsilon_s},$$

where each $\sigma_\epsilon \in \sum \mathbb{R}[X]^2$.

Schmüdgen's Theorem says that if K_S is compact then any $f \in \mathbb{R}[X]$ with $f > 0$ on K_S is in T_S. In this note we show that if the f and the generating polynomials $g_1, \ldots, g_s$ are in $\mathbb{Q}[X]$, then f has a representation in T_S in which all sums of squares σ_ϵ are in $\sum \mathbb{Q}[X]^2$. This follows from T. Wörmann's algebraic proof of the theorem using the classical abstract Positivstellensatz, and a generalization of Wörmann's crucial lemma due to M. Schweighofer.

The Abstract Positivstellensatz

The abstract Positivstellensatz is usually attributed to Kadison-Dubois, but now thought to be proven earlier by Krivine or Stone. For details on the history of the result, see [**?**, Section 5.6]. The setting is preordered commutative rings.

Let A be a commutative ring with $\mathbb{Q} \subseteq A$. A subset $T \subseteq A$ is a (proper) preordering if $T + T \subseteq T$, $T \cdot T \subseteq T$, and $-1 \notin T$. For $S = \{a_1, \ldots, a_k\} \subseteq A$, we define the preordering generated by S, T_S, exactly as for $A = \mathbb{R}[X]$.

Fix a preordered ring (A, T) and denote by Sper A the real spectrum of (A, T), i.e., the set of orderings of A which contain T. Then define $H(A) = \{a \in A \mid n \pm a \geq 0 \text{ on Sper } A \text{ for some } n \in \mathbb{N}\}$, the ring of geometrically bounded elements in (A, T), and $H'(A) = \{a \in A \mid n \pm a \in T \text{ for some } n \in \mathbb{N}\}$, the ring of arithmetically bounded elements in (A, T). Note that $H'(A) \subseteq H(A)$.

The following version of the abstract Positivstellensatz can be found in [**?**, Theorem 1]:

Theorem 1. *Given the preordered ring (A, T) as above and suppose $A = H'(A)$. For any $a \in A$, if $a > 0$ on Sper A, then $a \in T$.*

Schmüdgen's Theorem

Consider the case where $A = \mathbb{R}[X]$ and $T = T_S$ for $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{R}[X]$. Let $K = K_S$, then as is well known, K embeds densely in Sper A and hence $H(A) = \{f \in \mathbb{R}[X] \mid f \text{ is bounded on } S\}$. If S is compact, this implies $H(A) = A$ and Schmüdgen's Theorem follows from the following lemma, due to Wörmann [**?**]:

Lemma 1. *With A, T, and S as above, if $H(A) = A$ then $H'(A) = A$.*

Our result follows from a generalization of Lemma **??**, due to Schweighofer [**?**, Theorem 4.13]:

Theorem 2. *Let K be a subfield of $\mathbb{R}$ and (A, T) a preordered K-algebra such that $K \subseteq H'(A)$. Suppose A has finite transcendence degree over K. Then*

$$A = H(A) \Rightarrow A = H'(A).$$

Given $S = \{g_1, \ldots, g_s\} \subseteq \mathbb{Q}[X]$, let $T = T_S \subseteq \mathbb{Q}[X]$ and let $K = K_S \subseteq \mathbb{R}^n$, i.e., we define K over $\mathbb{R}$, not $\mathbb{Q}$. We are going to apply the theorem to the preordered ring $(\mathbb{Q}[X], T)$. Note that the condition $K \subseteq H'(A)$ holds in this case since $\mathbb{Q}^+ = \sum \mathbb{Q}^2$.

Theorem 3. *Suppose K is compact and $f \in \mathbb{Q}[X]$ such that $f > 0$ on K. Then $f \in T$.*

Proof. The argument is exactly that of Wörmann's proof of the Schmüdgen's Theorem above. Since K is compact, every element of $\mathbb{Q}[X]$ is bounded on K. Then K dense in Sper A implies that $H(\mathbb{Q}[X]) = \mathbb{Q}[X]$, hence by Theorem **??** we have $\mathbb{Q}[X] = H'(A)$. The result follows from Theorem **??**. □

References

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