Positive polynomials and the moment problem for cylinders with compact cross-section

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1 Introduction

Given a semialgebraic set K in $\mathbb{R}^n$ defined by finitely many polynomial inequalities $\{g_1 \geq 0, \ldots, g_s \geq 0\}$, $g_i \in \mathbb{R}[X] := \mathbb{R}[x_1, \ldots, x_n]$, let T be the preorder in $\mathbb{R}[X]$ generated by the g_i's. We consider three properties:

$$(\dagger) \qquad \forall f \in \mathbb{R}[X], \quad f > 0 \text{ on } K \Rightarrow f \in T.$$

$$(\ddagger) \qquad \forall f \in \mathbb{R}[X], \quad f > 0 \text{ on } K \Rightarrow \exists q \in T \text{ such that } \forall \text{ real } \epsilon > 0, f + \epsilon q \in T.$$

$$(*) \qquad \{g_1, \ldots, g_s\} \text{ solves the moment problem for } K$$

By the latter, we mean that the linear functionals on $\mathbb{R}[X]$ which come from integration with respect to a positive Borel measure on K are characterized as those which are non-negative on T. For details, see, e.g., [6].

Clearly, ($\dagger$) implies ($\ddagger$), and Kuhlmann and Marshall [2] have shown that ($\ddagger$) implies ($*$). Schmüdgen [8] showed that if K is compact, then ($*$) and ($\dagger$) hold, regardless of the choice of generators $\{g_i\}$. The proof of this result, which uses functional analysis, is not constructive. Recently, Schweighofer [10] has given a constructive proof of Schmüdgen's theorem with degree bounds on the output data.

If K is not compact, these properties do not hold in general and can depend on the choice of generators. Scheiderer [7] has shown that ($\dagger$) does not hold if K is not compact and $\dim K \geq 3$, or if $\dim K = 2$ and K contains a 2-dimensional cone. In [2] and [6], it is shown that if $\dim K \geq 2$ and K contains an open cone, then ($*$) does not hold.

In [6], the question of whether ($*$) holds is settled for closed semialgebraic subsets of smooth affine curves; roughly speaking, the answer depends on the behaviour of the real points at infinity. Finally, in [2], the case of non-compact closed semialgebraic subsets of $\mathbb{R}$ is settled. In this case, ($*$) and (†) are equivalent and hold only if a particular set of generators is chosen.

In this paper, we study these properties for the following general case, which is not covered above: **cylinders with compact cross-section**, i.e., closed semialgebraic sets of the form $K \times U$ where $K \subseteq \mathbb{R}^n$ is compact, and $U \subseteq \mathbb{R}$ is not compact. We extend the Schweighofer algorithm to show in this case that (†) holds for f with a certain boundedness property. As a corollary, we obtain property (‡) holds and hence ($*$). This settles Open Problem 1 in [2]. In [9], ($*$) is also proven in this case, using entirely different methods.

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2 Notation and Background

Fix $n \geq 1$ and let $\mathbb{R}[X]$ be the polynomial ring in n variables: $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$. We will write $\mathbb{R}[t]$ for the polynomial ring in one variable and $\mathbb{R}[X, t]$ for $\mathbb{R}[x_1, \ldots, x_n, t]$. $\sum \mathbb{R}[X]^2$ denotes the set of sums of squares in $\mathbb{R}[X]$ and we say f is **sos** if $f \in \sum \mathbb{R}[X]^2$. If $S = \{g_1, \ldots, g_s\}$ is a finite set of polynomials in k variables, let K_S denote the basic closed semialgebraic set in $\mathbb{R}^k$ generated by S, i.e.,

$$K_S = \{\alpha \in \mathbb{R}^k \mid g_1(\alpha) \geq 0, \ldots, g_s(\alpha) \geq 0\}.$$

Let T_S be the associated preorder in the appropriate real polynomial ring, i.e., T_S consists of finite sums of elements of the form

$$\sigma g_1^{\epsilon_1} \ldots g_s^{\epsilon_s},$$

where σ is sos and $\epsilon_i \in \{0, 1\}$.

In addition to the three properties above, we can consider

$$(\dagger') \qquad \forall f \in \mathbb{R}[X], \quad f \geq 0 \text{ on } K \Rightarrow f \in T.$$

In other words, we replace > 0 by ≥ 0 in the definition of (†). In general, (†') does not hold, even in the compact case. For example, in $\mathbb{R}[t]$, it is easy to see that $1 - t$ is not in the preorder generated by $\{(1 - t^2)^3\}$ even though $1 - t \geq 0$ on $[-1, 1] = K_{\{(1-x^2)^3\}}$.

The case of non-compact closed semialgebraic subsets of $\mathbb{R}$ has been settled completely by Kuhlmann and Marshall [2]. We recall their results.

Definition 1. Suppose $K \subseteq \mathbb{R}$ is a closed semialgebraic set. Define a set $S \subseteq \mathbb{R}[t]$ as follows:

1. If $a \in K$ and $(-\infty, a) \cap K = \emptyset$, then $t - a \in S$.
2. If $a \in K$ and $(a, \infty) \cap K = \emptyset$, then $a - t \in S$.
3. If $a, b \in K$, $a < b$, and $(a, b) \cap K = \emptyset$, then $(t-a)(t-b) \in S$
4. S contains no other elements

S is called the **natural set of generators for** K

It is easy to see that if K and S are as in the definition, then $K_S = K$. The following is [2, 2.1]:

Theorem 1. *Suppose $K \subseteq \mathbb{R}$ is closed semialgebraic, K is not compact, and $S \subseteq \mathbb{R}[t]$ is such that $K_S = K$. Then $(*)$ holds iff $(\dagger)$ holds iff S contains the natural set of generators for K. Furthermore, if $(\dagger)$ holds for S, then $(\dagger')$ holds also.*

3 Extending the Schweighofer algorithm

We want to study the properties $(\dagger)$, $(\ddagger)$, and $(*)$ for basic closed semialgebraic sets of the form $K_S \times K_U$, where $\emptyset \neq K_S \subseteq \mathbb{R}^n$ is compact and $K_U \subseteq \mathbb{R}$ is not compact. By Theorem 1 above, we will need to use the natural set of generators for K_U. We will show that $(\dagger)$ holds for all f which satisfy a certain boundedness condition and as a corollary, we obtain $(\ddagger)$ for all f and hence $(*)$ for all semialgebraic sets of this type.

Let us fix $S = \{g_1, \ldots, g_m\} \subseteq \mathbb{R}[X]$ such that $K_S \neq \emptyset$ is compact. Also, fix finite $U \subseteq \mathbb{R}[t]$ such that K_U is not compact and U contains the natural set of generators for K_U. Let $K := K_{S \cup U} = K_S \times K_U$ and let $T \subseteq \mathbb{R}[X, t]$ be the preorder generated by $S \cup U$.

For $b \in \mathbb{R}$, define $e_b : \mathbb{R}[X, t] \to \mathbb{R}[X]$ by $e_b(f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, b)$ and write f_b for $e_b(f)$. Given $f > 0$ on K, then for each $b \in K_U$, we have $f_b > 0$ on K_S. Since K_S is compact, we can apply the Schweighofer construction to find a representation of f_b in T_S. The idea is to "glue together" these representations in order to obtain a representation of f in T. To do this, we need a universal bound on the degree of the representation for all f_b's, which will require an additional assumption on f.

The central idea of the algorithm in the compact case is to reduce to the case of a homogeneous polynomial positive on a standard simplex and then apply Pólya's Theorem. In particular, a constructive version of Pólya's Theorem from [5] is used.

Definition 2. For $k \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, define

$$c(\alpha) := \frac{(\alpha_1 + \cdots + \alpha_k)!}{\alpha_1! \dots \alpha_k!}.$$

Given a polynomial g in k variables of degree d, let a_α denote the coefficient of g corresponding to the monomial with exponent α. Then set

$$L(g) := \max\left(\frac{a_\alpha}{c(\alpha)}\right),$$

where the max is taken over $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq d$. We call $\{\frac{a_\alpha}{c(\alpha)}\}$ the **normalized coefficients** of g.

The following version of Pólya's Theorem is [5, Theorem 1]:

Theorem 2. *Suppose that $F \in \mathbb{R}[X]$ is homogeneous of degree d and $F > 0$ on*

$$\Delta_n := \{(u_1, \dots, u_n) \in [0, \infty)^n \mid u_1 + \cdots + u_n = 1\}.$$

Then for $N \in \mathbb{N}$ such that

$$N > \frac{d(d-1)}{2} \frac{L(F)}{\min\{F(u) \mid u \in \Delta_n\}} - d,$$

$(x_1 + \cdots + x_n)^N F(X)$ has only positive coefficients.

Definition 3. Suppose $f \in \mathbb{R}[X, t]$ and $K \subseteq \mathbb{R}^n$. Let m be the maximum degree of f in t. We say f is **fully m-ic on** K if for all $u \in K$, $f(u, t)$ has degree m. In other words, if $h(X)t^m$ is the leading term of f as a polynomial in t, then f is fully m-ic on K iff $h(X)$ has no zeros in K.

Proposition 1. *Let $K_S \times K_U$ be as above and suppose $f \in \mathbb{R}[X, t]$ with $f > 0$ on $K_S \times K_U$. Let m be the degree of f in t and suppose f is fully m-ic on K_S. For each $b \in K_U$, set $L_b := L(f_b)$ and $\mu_b := \min\{f_b(u) \mid u \in K_S\}$.*

1. *There exists $g(t) \in \mathbb{R}[t]$ with $\deg g = m$ such that for all $b \in K_U$, $L_b \leq g(b)$.*
2. *$\dfrac{L_b}{\mu_b}$ and $\dfrac{g(b)}{\mu_b}$ are bounded on K_U.*

Proof. K_U contains $(-\infty, a]$ or $[a, \infty)$ for some a.
1. Assume K_U contain $[a, \infty)$. Write $f = \sum c(\alpha) a_{\alpha,j} X^\alpha t^j$, where the sum is over $\alpha \in \mathbb{N}^n$, $j \in \mathbb{N}$ with $|\alpha| + j \leq \deg f$. Then in f_b, the normalized coefficient of X^α is $\{\sum_j a_{\alpha,j} b^j\}$. Thus L_b is the maximum over α of $|\sum_j a_{\alpha,j} b^j|$. Since $a_{\alpha,j} = 0$ for $j > m$, for each α there exists $r(\alpha) \in \mathbb{R}$ such that for sufficiently large b, $|\sum_j a_{\alpha,j} b^j| \leq r(\alpha) b^m$. Then for some $r_1 \in \mathbb{R}$ and $w \in \mathbb{N}$, $L_b \leq r_1 b^m$ for $b \in K_U$,

$b > w$. If K_U does not contain an interval $(-\infty, a']$, then let $s = \max\{L_b \mid b \leq w\}$ and $g(t) = r_1 t^m + s$ satisfies $L_b \leq g(b)$ for all $b \in K_U$. If K_U does contain some $(-\infty, a']$, then m must be even and and for sufficiently large $|b|$, $|\sum_j a_{\alpha,j} b^j| \leq r(\alpha) b^m$. In this case, let $s = \max\{L_b \mid |b| \leq w\}$ and $g(t) = r_1 t^m + s$.

The proof for K_U containing $(-\infty, a]$ and not an interval unbounded from above is similar to the proof of the first case.

2. Again assume K_U contains $[a, \infty)$. Write f as a polynomial in t:

$$f(X,t) = h(X)t^m + \sum_{j<m} h_j(X)t^j.$$

Since $h(X)$ has no zeros on K_S, we must have $h(X) > u$ on K_S for some $u \in \mathbb{R}^+$. Also, since K_S is compact, for each $j < m$ there is $M_j \in \mathbb{N}$ such that $h_j(X) < M_j$ on K_S. Then, on K_S,

$$f_b(X) \geq u * b^m - \sum_j M_j b^j > r b^m$$

for some constant r and for b sufficiently large. Then, since $\deg g = m$, it follows easily that $\frac{g(b)}{\mu_b}$ is bounded. Finally, $L_b \leq g(b)$ for all $b \in K_U$ implies $\frac{L_b}{\mu_b}$ is bounded. If K_U contains only $(-\infty, a]$, then the proof is similar.

□

Our goal is to prove the following:

Theorem 3. *With K and T as above, property (†) holds for any $f \in \mathbb{R}[X,t]$ which is fully m-ic on K_S. In other words, for such f, $f > 0$ on K implies $f \in T$.*

As in [10], we make some convenient assumptions about S. First, we assume that $K_S \subseteq (-1,1)^n$; an easy scaling argument shows that this case implies the general case. Fix $\epsilon > 0$ so that $K_S \subseteq [-1+2\epsilon, 1-2\epsilon]^n$ and scale each g_i by a positive factor so that $2n\epsilon - (g_1 + \cdots + g_m) > 0$ on K_S. Now we define $M := 2n + m + 1$ polynomials $\{h_i\}$ in $\mathbb{R}[X]$ as follows:

$$\begin{aligned} h_1 &= 1 - \epsilon + x_1, \quad \ldots \quad , h_n = 1 - \epsilon + x_n, \\ h_{n+1} &= 1 - \epsilon - x_1, \quad \ldots \quad , h_{2n} = 1 - \epsilon - x_n, \\ h_{2n+1} &= g_1, \quad \ldots \quad , h_{2n+m} = g_m, \\ h_M &= 2n\epsilon - (g_1 + \cdots + g_m). \end{aligned}$$

Note that $\sum h_i = 2n$ and each h_i is in T_S: $h_1, \ldots, h_{2n}$ and h_M by Schmüdgen's Theorem and the remaining trivially. For ease of exposition, for $\beta \in \mathbb{N}^M$, we write H^β for $h_1^{\beta_1} \ldots h_M^{\beta_M}$. By the previous remark, for $a \in \mathbb{R}^+$ and any $\alpha \in \mathbb{N}^M$, $aH^\alpha \in T_S$.

Let $\mathbb{R}[Y]$ denote $\mathbb{R}[y_1, \ldots, y_M]$ and let $\mathbb{R}[Y,t]$ denote $\mathbb{R}[y_1, \ldots, y_M, t]$. Define $\phi : \mathbb{R}[Y] \mapsto \mathbb{R}[X]$ by $\phi(y_i) = h_i$ and $\bar{\phi} : \mathbb{R}[Y,t] \to \mathbb{R}[X,t]$ similarly (with $\bar{\phi}(t) = t$). We also have the maps e_b on $\mathbb{R}[Y,t]$ and $\mathbb{R}[X,t]$. These are all homomorphisms and it is easy to see that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}[Y,t] & \xrightarrow{\bar{\phi}} & \mathbb{R}[X,t] \\ {\scriptstyle e_b}\downarrow & & {\scriptstyle e_b}\downarrow \\ \mathbb{R}[Y] & \xrightarrow{\phi} & \mathbb{R}[X] \end{array}$$

Define $Z := \dfrac{y_1 + \cdots + y_M}{2n} \in \mathbb{R}[Y]$ and note that $Z \in \ker \phi$ and $\deg Z = 1$. Z is useful for homogenizing or raising the degree of a polynomial in $\mathbb{R}[Y]$ without changing its image under ϕ.

Here is a rough outline of the Schweighofer algorithm: Given $p > 0$ on K_S, construct a homogeneous $Q \in \mathbb{R}[Y]$ such that $\phi(Q) = p$ and $Q > 0$ on Δ_M. Using Theorem 2, find N so that $Z^N Q$ equals a polynomial with only positive coefficients. Then, applying ϕ to both sides of this equation, we obtain $p = \sum a_\alpha H^\alpha$ and hence a representation of p in T_S (modulo representations of the h_i's).

Definition 4. Given $g \in \mathbb{R}[X]$ with $\deg g = d$, write $g = G_0 + \cdots + G_d$, where G_i is the homogeneous part of g of degree i. For any $k \geq d$, define homogeneous $P^{(k)}(g) \in \mathbb{R}[Y]$ of degree k by

$$P^{(k)}(g) := \sum_{i=0}^{d} G_i \left(\frac{1}{2} y_1 - \frac{1}{2} y_{n+1}, \ldots, \frac{1}{2} y_{n+1} - \frac{1}{2} y_{2n} \right) \cdot Z^{k-i}$$

Note that for all $k \geq d$, $\phi(P^{(k)}(g)) = g$. The construction now proceeds by adding an element of $\ker \phi$ to some $P^{(k)}(p)$ in order to make it positive on Δ_M. We need to extract this part of the construction; the next result and its proof are completely contained in the proof of [10, Lemma 9].

Lemma 1. *We make the assumptions and definitions as above for compact K_S. Then there are constants $1 \leq c_0, c_2$ and a homogeneous polynomial $R_0 \in \ker \phi$ of degree d_0 such that the following holds: Given $d \geq 1$ and suppose $p \in \mathbb{R}[X]$ with $\deg p = d$ such that $L(p) = 1$ and $p > 0$ on K_S. Let $l = \max\{d_0, d\}$, let μ be the minimum of p on K_S, and set $R = R_0 \cdot Z^{l-d_0}$. Then for*

$$\lambda = c_2 d^2 n^d \left(\frac{d^2 n^d}{\mu} \right)^{c_0},$$

we have

$$P^{(l)}(p) + \lambda R \geq \frac{\mu}{2(2n)^l} \text{ on } \Delta_M$$

We need two generalizations of the lemma, which are easily obtained:

Corollary 1. *We make all the assumptions and definitions as in Lemma 1, except we only assume* $1 \le \deg p \le d$. *Then the conclusion of Lemma 1 holds.*

Proof. Let $u = \deg p$, if we apply Lemma 1 to p we obtain

$$P^{(\tilde{l})}(p) + \tilde{\lambda} R \ge \frac{\mu}{2(2n)^l},$$

where $\tilde{l} = \max\{d_0, u\} \le l$ and

$$\tilde{\lambda} = c_2 u^2 n^u \left(\frac{u^2 n^u}{\mu} \right)^{c_0}.$$

It is easy to see that $P^{(l)}(p) = P^{(\tilde{l})}(p) \cdot Z^{l-\tilde{l}}$ and $\lambda \ge \tilde{\lambda}$. This implies $P^{(l)}(p) + \lambda R \ge P^{(\tilde{l})}(p) + \tilde{\lambda} R \ge \dfrac{\mu}{2(2n)^l}$ on Δ_M. ☐

We need the corollary without the assumption that $L(p) = 1$.

Corollary 2. *We make the assumptions and definitions as above for compact* K_S. *Then there are constants* $1 \le c_0, c_2$, *and a homogeneous polynomial* $R_0 \in \ker \phi$ *of degree* d_0 *such that the following holds: Given* $d \ge 1$ *and suppose* $p \in \mathbb{R}[X]$ *of degree* $\le d$ *with* $p > 0$ *on* K_S. *Let* $l = \max\{d_0, d\}$, $\mu = \min\{p(u) \mid u \in K_S\}$ *and set* $R = R_0 \cdot Z^{l-d_0}$. *Then for*

$$\lambda = c_2 d^2 n^d \left(d^2 n^d \frac{L(p)}{\mu} \right)^{c_0},$$

we have

$$P^{(l)}(p) + L(p) \cdot \lambda \cdot R \ge \frac{\mu}{2(2n)^l} \quad \text{on } \Delta_M$$

Proof. Let $p' = \dfrac{p}{L(p)}$, then obviously $L(p') = 1$. It is easy to see that $P^{(k)}(p') = \dfrac{P^{(k)}(p)}{L(p)}$ and the minimum of p' on K_S is $\dfrac{\mu}{L(p)}$. Applying Corollary 1, we obtain

$$\frac{P^{(k)}(p)}{L(p)} + \lambda R > \frac{1}{L(p)} \frac{\mu}{2(2n)^l}$$

on Δ_M and, multiplying by $L(p)$, we obtain the desired result. ☐

Proof of Theorem 3: We are given $f \in \mathbb{R}[X,t]$ with $f > 0$ on K such that $\deg_t f = m$ and f is fully m-ic on K_S. Let d be the maximum degree in X of f. For each $b \in K_U$, let μ_b denote the minimum of f_b on K_S and write L_b for $L(f_b)$. Let c_0, c_2, R_0 and d_0 be as in Corollary 2 and set $l = \max\{d_0, d\}$ and $R = R_0 \cdot Z^{l-d_0}$.

Decompose $f = F_0 + \cdots + F_d$, where F_i is the part of f which is degree i in X. Define $Q \in \mathbb{R}[Y, t]$ by

$$Q = \sum_{i=0}^{d} F_i \left(\frac{1}{2} y_1 - \frac{1}{2} y_{n+1}, \ldots, \frac{1}{2} y_n - \frac{1}{2} y_{2n}, t \right) \cdot Z^{l-i}$$

Note that $e_b(F_i)$ is the degree i part of f_b (or zero if there is no degree i part), hence $e_b(Q) = P^{(l)}(f_b)$. Also, $\bar{\phi}(Q) = \sum F_i(x_1, \ldots, x_n, t) = f$.

By Proposition 1, there exists $g(t) \in \mathbb{R}[t]$, $\deg g = m$, so that $L_b \leq g(b)$ for all $b \in K_U$. Also by the proposition, we can find $W \in \mathbb{N}$ such that for all $b \in K_U$, $\frac{L_b}{\mu_b} \leq W$ and $\frac{g(b)}{\mu_b} \leq W$. Let

$$\lambda = c_2 d^2 n^d \left(d^2 n^d W \right)^{c_0}$$

and define

$$\tilde{Q} := Q + g(t) \cdot \lambda \cdot R.$$

Write $\tilde{Q}_b$ for $e_b(\tilde{Q})$, then $\bar{\phi}(\tilde{Q}) = \bar{\phi}(Q) = f$ and $\tilde{Q}_b = P^{(l)}(f_b) + g(b) \cdot \lambda \cdot R$.

For each $b \in K_U$, let

$$\lambda_b = c_2 d^2 n^d \left(d^2 n^d \frac{L_b}{\mu_b} \right)^{c_0},$$

Note that $\lambda \geq \lambda_b$ for all b.

Applying Corollary 2 for each b, we then have

$$P^{(l)}(f_b) + L_b \cdot \lambda_b \cdot R \geq \frac{\mu_b}{2(2n)^l} \text{ on } \Delta_M \tag{1}$$

Since $\lambda_b \leq \lambda$ and $L_b \leq g(b)$, (1) implies

$$\tilde{Q}_b = P^{(l)}(f_b) + g(b) \cdot \lambda \cdot R \geq \frac{\mu_b}{2(2n)^l} \text{ on } \Delta_M \tag{2}$$

Claim 1: There exists $N \in \mathbb{N}$ so that for each $b \in K_U$, $(y_1 + \cdots + y_M)^N \tilde{Q}_b$ has only positive coefficients.

Proof of claim: By Theorem 2, $(\sum y_i)^{N_b} \tilde{Q}_b$ has only positive coefficients for

$$N_b \geq \frac{l(l-1)}{2} \cdot \frac{L(\tilde{Q}_b)}{\min\{\tilde{Q}_b(u) \mid u \in \Delta_M\}}.$$

From the proof of [10, Lemma 9], we have $L(P^{(l)}(f_b)) \leq \frac{(d+1)}{2^l} L_b$ and $L(R) \leq \frac{L(R_0)}{(2n)^{l-d_0}}$, hence

$$L(\tilde{Q}_b) \leq \frac{d+1}{2^l} L_b + g(b) \cdot \lambda \cdot \frac{L(R_0)}{(2n)^{l-d_0}}.$$

By (2), the minimum of $\tilde{Q}_b$ on Δ_M is $\geq \frac{\mu_b}{(2n)^{l-d_0}}$. Recall we have $\frac{L_b}{\mu_b} \leq W$ and $\frac{g(b)}{\mu_b} \leq W$, hence

$$\begin{aligned} \frac{L(\tilde{Q}_b)}{\min\{\tilde{Q}_b(u) \mid u \in \Delta_M\}} &\leq \frac{(2n)^{l-d_0}}{\mu} \left(\frac{d+1}{2^l} L_b + g(b) \cdot \lambda \cdot \frac{L(R_0)}{(2n)^{l-d_0}} \right) \\ &\leq W \left(\frac{n^l}{(2n)^{d_0}} (d+1) + \lambda L(R_0) \right) \end{aligned}$$

This implies that if $N \in \mathbb{N}$ with

$$N \geq \frac{l(l-1)}{2} W \left(\frac{n^l}{(2n)^{d_0}} (d+1) + \lambda \cdot L(R_0) \right),$$

then $N_b \leq N$ and the claim holds.

Consider $(\sum y_i)^N \tilde{Q} \in \mathbb{R}[Y, t]$ and write this as a polynomial in $y_1, \ldots, y_M$ with coefficients in $\mathbb{R}[t]$:

$$(3) \qquad (\sum y_i)^N \tilde{Q} = \sum_{\alpha \in \mathbb{R}^M} A_\alpha(t) Y^\alpha.$$

Applying $\bar{\phi}$ to both sides yields an expression

$$(4) \qquad (2n)^N f = \sum_{\alpha \in \mathbb{R}^M} A_\alpha(t) \cdot H^\alpha.$$

Claim: For each α, $A_\alpha(b) > 0$ for all $b \in K_U$.
Proof of claim: By the previous claim, $(y_1 + \cdots + y_M)^N \tilde{Q}_b$ has only positive coefficients. Applying e_b to both sides of (3) yields $(\sum y_i)^N \tilde{Q}_b = \sum A_\alpha(b) \cdot Y^\alpha$, which implies $A_\alpha(b) > 0$ for each α.

Since $A_\alpha > 0$ on K_U, by Theorem 1, A_α is in T_U, the preorder in $\mathbb{R}[t]$ generated by U. Substituting representations of the A_α's in T_U into (4) yields a representation of f in T, proving Theorem 3. □

Corollary 3. *Given the above notations and assumptions. Then (‡) holds for K and T, i.e., given $f > 0$ on $K_S \times K_U$, there exists $q \in T$ such that for all $\epsilon > 0$, $f + \epsilon q \in T$.*

Proof. Assume f has degree m in t and let $q = t^{2m}$. Clearly, $f + \epsilon q$ is fully $2m$-ic on K_U. Therefore, we are done by Theorem 3. □

Theorem 4. *Let K, T be as above. Then property (∗) holds, i.e., $S \cup U$ solves the moment problem for K.*

Remark 1. Theorem 4 is also proven in [3] using different methods.

References

[1] T. Jacobi and A. Prestel, *Distinguished representations of strictly positive polynomials*, J. reine angew. Math. **532** (2001), 223-235.

[2] S. Kuhlmann and M. Marshall, *Positivity, sums of squares, and the multidimensional moment problem*, Trans. Amer. Math. Soc. **354** (2002), 4285-4301.

[3] S. Kuhlmann, M. Marshall, and N. Schwartz, *Positivity, sums of squares, and the multidimensional moment problem II*, preprint.

[4] G. Pólya, *Über positive Darstellung von Polynomen Vierteljschr*, Naturforsch. Ges. Zürich **73** (1928), 141-145, in *Collected Papers* **2** (1974), MIT Press, 309-313.

[5] V. Powers and B. Reznick, *A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra*, J. Pure and Applied Alg. **164** (2001), 221-229.

[6] V. Powers and C. Scheiderer, *The moment problem for non-compact semialgebraic sets*, Adv. Geo. **1** (2001), 71-88.

[7] C. Scheiderer, *Sums of squares of regular functions on real algebraic varieties*, Trans. Amer. Math. Soc. **352** (1999), 1030-1069.

[8] K. Schmüdgen, *The K-moment problem for compact semialgebraic sets*, Math. Ann. **289** (1991), 203-206.

[9] K. Schmüdgen, *On the moment problem of closed semialgebraic sets*, preprint.

[10] M. Schweighofer, *On the complexity of Schmüdgen's Positivstellensatz*, preprint.