Polynomials Positive on Unbounded Rectangles

Victoria Powers *
Department of Mathematics and Computer Science,
Emory University,
Atlanta, GA 30322

Bruce Reznick †
Department of Mathematics
University of Illinois at Urbana-Champaign,
Urbana, IL 61801

April 30, 2004

1 Introduction and notation

Given a semialgebraic set $K \subseteq \mathbb{R}^N$ determined by a finite set of polynomial inequalities $\{g_1 \geq 0, \ldots, g_k \geq 0\}$, we want to characterize a polynomial f which is positive (or non-negative) on K in terms of sums of squares and the polynomials g_i used to describe K. Such a representation of f is an immediate witness to the positivity condition. Theorems about the existence of such representations also have various applications, notably in problems of optimizing polynomial functions on semialgebraic sets.

In case K is compact, Schmüdgen has proved that any polynomial which is positive on K is in the preorder generated by the g_i's, i.e., the set of finite sums of elements of the form $s_e g_1^{e_1} \ldots g_k^{e_k}, e_i \in \{0, 1\}$, where each s_e is a sum of squares of polynomials. Putinar has proved that, under certain conditions, the preorder can be replaced by the quadratic module, which is the set of sums $\{s_0 + s_1 g_1 + \cdots + s_k g_k\}$, where each s_i is a sum of squares. Using this result, Lasserre has developed algorithms for finding the minimum of a polynomial on such compact K, which transforms this into a semidefinite programming problem.

What happens when K is not compact? Scheiderer has shown that if K is not compact and $\dim K \geq 3$, then Schmüdgen's characterization can never hold,

---

*This material is based in part on work of this author performed while she was a visiting professor at Universidad Complutense, Madrid, Spain with support from the D.G.I. de España.

†This material is based in part upon work of this author, supported by the USAF under DARPA/AFOSR MURI Award F49620-02-1-0325. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of these agencies.

regardless of the g_i's chosen to describe K. Kuhlman and Marshall settled the case where K is not compact and contained in $\mathbb{R}$; the answer here depends on chosing the "right" set of generators for K. In this paper we consider some variations on these themes: we look at some canonical non-compact sets in $\mathbb{R}^2$ which are products of intervals and at some stronger and weaker versions of positivity.

We introduce some basic notation. Let $S = \{g_1, \dots, g_s\}$ denote a finite set of polynomials in $R_n := \mathbb{R}[x_1, \dots, x_n]$, and let

$$K = K_S = \bigcap_j \{a \in \mathbb{R}^n \mid f_j(a) \geq 0\}.$$

Write $\sum R_n^2$ for the set of finite sums of squares of elements of R_n; clearly, any $\sigma \in \sum R_n^2$ takes only non-negative values on $\mathbb{R}^n$. We shall say that an element of $\sum R_n^2$ is *sos*. The *preorder* generated by S, denoted T_S, is the set of finite sums of the type $\sum \sigma g_1^{\epsilon_1} \dots g_s^{\epsilon_s}$ where σ is sos and $\epsilon_i \in \{0, 1\}$. That is, a typical element of T_S has the shape

$$\sigma_0 + \sum_I \sigma_I \left(\prod_{i \in I} g_I \right),$$

where the sum is taken over all non-empty $I \subseteq \{1, \dots, s\}$, and each σ_- is sos. An important subset of the preorder is the *quadratic module*, M_S, which consists of sums in which $\sum_i \epsilon_i \leq 1$ for each summand. That is, a typical element of M_S has the shape

$$\sigma_0 + \sum_{k=1}^s \sigma_k g_k.$$

Clearly, $M_S \subseteq T_S$, and if $s \geq 2$, then inclusion is formally strict. However, there can be non-trivial equality. For example, if $S = \{1 - x, 1 + x\}$, then the identity

$$(1 + x)(1 - x) = \left(\frac{(1 - x)^2}{2} \right) (1 + x) + \left(\frac{(1 + x)^2}{2} \right) (1 - x) \tag{1}$$

shows that $(1 + x)(1 - x)$ is already in M_S, so $M_S = T_S$ for this case.

Various notions of positivity. For $K \subseteq \mathbb{R}^n$ and $f \in R_n$, we write $f \geq 0$ on K if $f(x) \geq 0$ for all $x \in K$ and $f > 0$ on K if $f(x) > 0$ for all $x \in K$. We consider a stronger version of positivity which considers positivity at "points at infinity". (This definition appeared in [13, Ch. 7], in the context of moment and quadrature problems.)

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, write $(x, 1)$ for $(x_1, \dots, x_n, 1) \in \mathbb{R}^{n+1}$, and let

$$x^* := \frac{(x, 1)}{|(x, 1)|} \in S^n \subset \mathbb{R}^{n+1}.$$

Suppose $K \subseteq \mathbb{R}^n$ is a closed set. Let $K^* = \{x^* \mid x \in K\}$ and let $\overline{K^*}$ be the closure of $\{x^* \mid x \in K\}$. For example, if $K = \mathbb{R}^2$, then K^* consists of the Northern Hemisphere, and $\overline{K^*}$ is the Northern Hemisphere plus the equator.

Suppose $p \in R_n$ of degree d and let $p^* \in R_{n+1}$ be the homogenization of p, i.e.,

$$p^*(x_1, \dots, x_{n+1}) := x_{n+1}^d \; p\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right).$$

For $x \in \mathbb{R}^n$, let $\Phi_{n,d}(x_1, \dots, x_n) := (1 + \sum_{i=1}^n x_i^2)^{d/2} = |(x,1)|^d$. It follows from homogeneity that

$$p(x) = p^*(x,1) = \Phi_{n,d}(x) p^*(x^*). \tag{2}$$

We say p is *projectively positive on* K if p^* is positive on $\overline{K^*}$, and write $p \gg 0$ on K. Clearly, $p \gg 0$ on $K \Rightarrow p > 0$ on K. A simple example shows that the converse is false: Let $K = \mathbb{R}^2$ and $p(x,y) = x^2y^2 + 1$, so that $p^*(x,y,z) = x^2y^2 + z^4$. Then $p > 0$ on K; but $p^*(1,0,0) = 0$, so p is not projectively positive on K. Observe that $\overline{K^*}$ is compact, and so if $p \gg 0$ on K, then p^* achieves a positive minimum on $\overline{K^*}$.

Proposition 1. *Suppose $K \subseteq \mathbb{R}^n$ is closed and $p \in R$ of degree d.*

(i) There exists $c > 0$ so that $p - c\,\Phi_{n,d} \geq 0$ on K iff $p \gg 0$ on K.

(ii) If K is compact, then $p \gg 0$ on K iff $p > 0$ on K.

(iii) If $(x_1, \dots, x_{n+1}) \in \overline{K^*} \setminus K^*$, then $x_{n+1} = 0$.

Proof. By (2), we have $p^* \geq c > 0$ on $\overline{K^*}$ if and only if

$$p(x) = \Phi_{n,d}(x) p^*(x^*) \geq c\,\Phi_{n,d}(x).$$

for $x \in K$, proving (i). For (ii), it suffices to show that $p > 0$ on K implies $p \gg 0$ on K. Since $\Phi_{n,d}$ is bounded (say, by M) on the compact set K, $p \geq c$ on K implies that $p^* \geq c/M$ on $\overline{K^*}$. Finally, suppose

$$(x_1, \dots, x_{n+1}) = \lim_{N \to \infty} (x_1^{(N)}, \dots, x_{n+1}^{(N)}),$$

where $(x_1^{(N)}, \dots, x_{n+1}^{(N)}) \in K^*$ and $x_{n+1} > 0$. Then $x_{n+1}^{(N)} > 0$ for $N \geq N_0$, and for each such N,

$$\left(\frac{x_1^{(N)}}{x_{n+1}^{(N)}}, \dots, \frac{x_n^{(N)}}{x_{n+1}^{(N)}}\right)$$

belongs to K. Since K is closed, the limit is in K, and by retracing the definition, we see that $(x_1, \dots, x_{n+1}) \in \overline{K^*}$. □

Fix S, and let $K = K_S$, $M = M_S$ and $T = T_S$. We consider six properties of S:

$$
\begin{array}{ll}
(*) & f \gg 0 \text{ on } K \Rightarrow f \in T \\
(*)_M & f \gg 0 \text{ on } K \Rightarrow f \in M \\
(**) & f > 0 \text{ on } K \Rightarrow f \in T \\
(**)_M & f > 0 \text{ on } K \Rightarrow f \in M \\
(***) & f \geq 0 \text{ on } K \Rightarrow f \in T \\
(***)_M & f \geq 0 \text{ on } K \Rightarrow f \in M
\end{array}
$$

There is an immediate diagram of implications:

$$
\begin{array}{ccccc}
(***) & \Rightarrow & (**) & \Rightarrow & (*) \\
\Uparrow & & \Uparrow & & \Uparrow \\
(***)_M & \Rightarrow & (**)_M & \Rightarrow & (*)_M
\end{array}
$$

In case $s = 1$, the two rows of properties coalesce; if K is compact, then the last two columns coalesce.

We summarize what is known about these properties: Schmüdgen's Theorem [16] says that if K is compact, then $(**)$ holds, regardless of the choice of S. Also, when K is compact, Putinar [12] has shown that $(**)_M$ holds iff M contains a polynomial of the form $r - \sum x_i^2$ for some non-negative r.

On the other hand, Scheiderer [15] has shown that if K is not compact and $\dim K \geq 3$, or if $\dim K = 2$ and K contains a two-dimensional cone, then $(**)$ fails. (Observe that K contains a two-dimensional cone iff $\overline{K^*}$ contains an arc on the equator of the unit sphere.) The proof is non-constructive. The case of non-compact semialgebraic subsets of $\mathbb{R}$ has been settled completely by Kuhlmann and Marshall [3]. They show that in this case, $(**)$ and $(***)$ are equivalent and hold iff S contains a specific set of polynomials which generate S (what they call the "natural set of generators"). They also show that $(**)_M$ and $(***)_M$ are equivalent, and only hold in a few special cases.

In general, $(***)$ will not hold, even in the compact case. An easy example is given by $S = \{(1 - x^2)^3\}$, in which case $K_S = [-1, 1]$ but $1 - x^2 \notin T_S$. See [17] for details on this example.

The authors [10] previously considered two special cases in $\mathbb{R}$, in which $K = [-1, 1]$ or $[0, \infty)$. (It is easily shown via linear changes of variable that the case of closed intervals in $\mathbb{R}$ reduces to one of $\{[-1, 1], [0, \infty), \mathbb{R}\}$.) For each of these intervals, $(***)_M$ has been long known, for the "natural set of generators". Hilbert knew (and saw no need to prove) that if $f(x) \geq 0$ for $x \in \mathbb{R}$, then f is a sum of (two) squares of polynomials; this corresponds precisely to $T_\emptyset$. If we take $S = \{1 + x, 1 - x\}$, so that $K = [-1, 1]$, then Bernstein proved $(**)$ in 1915. On the other hand, if $S = \{1 - x^2\}$, then Fekete proved $(***)$ (some time before 1930, the first reference seems to be [8]). The authors showed as Corollary 14 in [10] that if S is given and $K = [0, \infty)$, then $\epsilon + x \in T_S$ for all ϵ iff S contains cx for some $c > 0$. In other words, $(**)$ fails for the (non-compact) set $[0, \infty)$ unless the natural generator is included.

In view of the foregoing, this paper considers products of intervals in the plane. There are, up to linear changes and permutation of the variables, six cases of products of closed intervals in the plane, and we take the natural set of generators:

$$
\begin{array}{ll}
K_0 = [-1, 1] \times [-1, 1] & S_0 = \{1 - x, 1 + x, 1 - y, 1 + y\}; \\
K_1 = [-1, 1] \times [0, \infty) & S_1 = \{1 - x, 1 + x, y\}; \\
K_2 = [-1, 1] \times (-\infty, \infty) & S_2 = \{1 - x, 1 + x\}; \\
K_3 = [0, \infty) \times [0, \infty) & S_3 = \{x, y\}; \\
K_4 = [0, \infty) \times (-\infty, \infty) & S_4 = \{x\}; \\
K_5 = (-\infty, \infty) \times (-\infty, \infty) & S_5 = \emptyset.
\end{array}
$$

Since K_0 is compact, $f > 0$ and $f \gg 0$ are equivalent and $(**)$ holds. By the Putinar result, $(**)_M$ holds in this case as well. Scheiderer recently showed [15] that $(***)$ holds for K_0. Thus all but the possibly $(* * *)_M$ hold hold for K_0.

All properties fail for K_5, by classical results of Hilbert and Robinson. K_3 and K_4 contain two-dimensional cones, so Scheiderer's work implies that $(**)$ fails for them; we shall present simple examples in the next section. In fact, we will show that $(*)$ does not hold in these cases. Thus all properties fail for K_3 and K_4.

Finally, we consider K_1 and K_2. We show that $(**)_M$ does not hold for K_1 and that $(*)$ holds for K_2. It is still an open question whether or not $(**)$ holds for K_1 or K_2.

Projective positivity and optimization. Recently, there has been interest in using representation theorems such as those of Schmüdgen and Putinar for developing algorithms for optimizing polynomials on semialgebraic sets. Lasserre [4] [5] describes a method for finding a lower bound for the minimum of a polynomial on a basic closed semialgebraic set and shows that the method produces the exact minimum in the compact case. Marshall [6] shows that in the presence of a certain stability condition, the general problem can be reduced to the compact case, and hence can be handled using Lasserre's method. It turns out that Marshall's stability condition is intimately related to projective positivity.

Definition 1 (Marshall). Suppose $S = \{g_1, \ldots, g_s\} \subseteq R_n$ and $f \in R_n$ is bounded from below on K_S. We say f is *stably bounded from below on* K_S if for any $h \in R_n$ with $\deg h \leq \deg f$, there exists $\epsilon > 0$ so that $f - \epsilon h$ is also bounded from below on K_S.

Theorem 1 (Marshall). *Suppose S is given as above and f is stably bounded from below on K_S. Then there is a computable $\rho > 0$ so that the minimum of f on K_S occurs on the (compact) semialgebraic set $K_S \cap \{x \mid \rho - ||x||^2 \geq 0\}$.*

We now interpret Proposition 1 in terms of projective positivity.

Proposition 2. *Given $S = \{g_1, \ldots, g_s\}, f \subseteq R_n$. Then $f \gg 0$ on K_S implies f is stably bounded from below by 0 on K_S.*

Proof. By Proposition 1, $f \gg 0$ iff there is $c \in \mathbb{R}^+$ so that $f - c\Phi(n,d)(x) > 0$ on K_S. Given $h \in R_n$ with $\deg h = d$, then there is some $N > 0$ and $\epsilon > 0$ such that $\epsilon p(x) < c\Phi(n,d)(x)$ for $||x|| > N$. Then $f - \epsilon p > 0$ on $K_S \cap \{x \mid ||x|| > N\}$ and this implies $f - \epsilon p$ is bounded from below on K_S. □

Thus for applications to optimization, projective positivity is the "right" notion of positivity to consider. As Marshall remarks in [6]: "In cases where f is not stably bounded from below on K_S, any procedure for approximating the minimum of f using floating point computations involving the coefficients is necessarily somewhat suspect."

2 The plane, half plane, and quarter plane

In the section we consider the semialgebraic sets K_3, K_4, and K_5 with generators S_3, S_4, and S_5. As stated above, it has been shown that $(**)$ holds neither for K_5 (Hilbert) nor for K_3 and K_4 (Scheiderer). In this section, we will construct explicit examples showing that $(*)$ does not hold, which implies $(**)$ does not hold.

First we consider polynomials $f \in R_2 := R = \mathbb{R}[x,y]$ which are non-negative in the plane and review some results about when they are in ΣR^2. We shall use the standard terminology that p is *psd* if $p \geq 0$ on $\mathbb{R}^2$ and p is *pd* if $p > 0$ on $\mathbb{R}^2$ In 1888, Hilbert [2] gave a construction of a non-sos polynomial which is psd on $\mathbb{R}^2$. This construction was not explicit, and the first explicit example was found by Motzkin [7] in 1967. R. M. Robinson simplified Hilbert's construction [14]; we will use this example to construct the counterexamples in this section:

$$Q(x,y,z) = x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2.$$

For $a \geq 0$, let $Q_a(x,y,z) = Q(x,y,z) + a(x^2+y^2+z^2)^3$, then since Q is psd, Q_a is also pd for $a > 0$. It is shown in [14] (and the observation really goes back to [2]) that the cone of sos ternary sextic forms is closed. Since Q does not belong to this cone, it follows that for *some* positive value of a, Q_a is not sos. In fact, the methods of [1] can be used to show that Q_a is pd but not sos for $a \in (0, 1/48)$; we omit the details. Let $q_a(x,y) \in \mathbb{R}[x,y]$ be the dehomogenization of Q_a, then for $0 < a < 1/48$, q_a is pd and not sos. As already noted, $\overline{K_5^*}$ is the Northerm Hemisphere plus the equator, and $q_a^* = Q_a$, hence $q_a \gg 0$ on K_5 and q_a is not in T_5. Thus $(*)$ does not hold for K_5.

Note that Q is even in x and so we can consider $f(x,y) = q_a(\sqrt{x}, y)$, so that $f(x^2,y) = q_a(x,y)$. Then $f^*(x^2,y,z) = Q_a(x,y,z)$, hence $f^*(x,y,z) \geq 0$ for $x > 0$.

It is easy to see that $\overline{K_4^*}$ is the quarter sphere plus half the equator; thus $f \gg 0$ on K_4. But if $f \in T_4$, then there exist sos σ_j so that

$$f(x,y) = \sigma_0(x,y) + x\sigma_1(x,y).$$

If we replace x by x^2 above, we obtain

$$Q_a(x,y) = f(x^2,y) = \sigma_0(x^2,y) + x^2\sigma_1(x^2,y).$$

This implies that Q_a is sos, a contradiction.

A virtually identical argument shows that $q_a(\sqrt{x},\sqrt{y}) \gg 0$ on K_3 for $a > 0$, but does not belong to T_3.

3 Non-compact strips in the plane

Before we discuss K_1 and K_2, we make a detour to $K = [-1,1]$. There are two natural sets of generators for K. Let $S_1 = \{1-x, 1+x\}$ and $S_2 = \{1-x^2\}$. Then clearly $K_{S_1} = K_{S_2} = K$ and $M_{S_2} = T_{S_2}$, because $|S_2| = 1$. As remarked earlier, (1) implies that $M_{S_1} = T_{S_1}$; finally, $T_{S_2} \subseteq T_{S_1}$ is immediate and

$$1 \pm x = \frac{(1 \pm x)^2}{2} + \frac{(1-x^2)}{2} \tag{3}$$

shows the converse. Thus it does not matter whether one takes S_1 or S_2 (or M or T) in discussing $[-1,1]$.

What do (1) and (3) have to say in the plane? First, for K_2, we might take either S_1 or S_2 above as the set of generators, keeping in mind that the set of possible σ's is taken from $\sum R_2^2$, rather than $\sum R_1^2$ as above. Then, once again M and T are not affected by the choice of generators and $M = T$. For K_1, we similarly have from (3) that $T_{\{1-x^2,y\}} = T_{\{1-x,1+x,y\}}$ and $M_{\{1-x^2,y\}} = M_{\{1-x,1+x,y\}}$. However, in this case, $T \neq M$. In fact, $y(1-x)$, which evidently is an element of $T_{\{1-x,1+x,y\}}$, does not belong to $M_{\{1-x,1+x,y\}} = M_{\{1-x^2,y\}}$.

Theorem 2. *Suppose $S = \{f_1(x), \dots, f_m(x), y\}$ is such that $K_S = K_1$. Then for every $f(x) \in \mathbb{R}[x]$, we have $g(x,y) = f(x) + y(1-x) \notin M_S$. In particular, $(**)_M$ does not hold for S.*

Proof. We show that that there cannot exist an identity

$$g(x,y) = f(x) + y(1-x) = \sigma_0(x,y) + \sum_{i=1}^{m} \sigma_i(x,y) f_i(x) + \sigma_{m+1}(x,y) \cdot y, \tag{4}$$

where the σ_i's are sos. Suppose (4) holds, and let

$$I = \{a \in [0,1) \mid \prod_i f_i(a) \neq 0\};$$

I is the interval $[0,1)$ minus a finite set of points. Fix $a \in I$. Since $(a,y) \in K_1$, it follows that $f_i(a) > 0$. Consider (4) when $x = a$:

$$f(a) + y(1-a) = \sigma_0(a,y) + \sum_{i=1}^{m} \sigma_i(a,y) f_i(a) + \sigma_{m+1}(a,y) \cdot y. \qquad (5)$$

Each $\sigma_i(a,y)$ is sos, and hence psd, and so as a polynomial in y has leading term $c_i y^{2m_i}$, where $c_i > 0$. Let $M = \max m_i$. Then the highest power of y occurring in any term on the right hand side of(5) is y^{2M} or y^{2M+1}, with positive coefficient or coefficients, and so no cancellation occurs. In view of the left hand side, this highest power must be y^1. It follows that $M = 0$, so that each $\sigma_i(a,y)$ is a constant. Writing $\sigma_i(x,y) = \sum_j A_{i,j}^2(x,y)$, we see that, $\deg_y A_{i,j}(a,y) = 0$ for $a \in I$. Suppose now that $\deg_y A_{i,j}(x,y) = m_{i,j}$ and write

$$A_{i,j}(x,y) = \sum_{k=0}^{m_{i,j}} B_{i,j,k}(x) y^k,$$

We have seen that $B_{i,j,k}(a) = 0$ for $a \in I$ if $k \geq 1$. Any polynomials which vanishes on I must be identically zero, hence $B_{i,j,k}(x) = 0$ for $k \geq 1$. Thus $m_{i,j} = 0$ and each $A_{i,j}(x,y)$ is, in fact, a polynomial in x alone, so that $\sigma_j(x,y) = \sigma_j(x)$. Therefore, (5) becomes

$$f(x) + y(1-x) = \sigma_0(x) + \sum_{j=1}^{m} \sigma_j(x) f_j(x) + y\sigma_{m+1}(x).$$

Taking the partial derivative of both sides of this equation with respect to y, we see that $1 - x = \sigma_{m+1}(x)$. This contradicts the assumption that σ_{m+1} is sos.

Let $f(x) = \epsilon$ for some $\epsilon > 0$. Then $g(x,y) = \epsilon + y(1-x)$ is positive on K_1, but $g \notin M$, thus $(**)_M$ fails for K_1. Observe, however, that if we take either of the standard generators for K_1, then

$$g(x,y) = \epsilon + y(1-x) = \epsilon + y \cdot \frac{(1-x)^2}{2} + \frac{y(1-x^2)}{2} \in T_S.$$

This shows that $T_S \neq M_S$ in this case. (The preceding construction works for any polynomial f which is positive on $[-1,1]$.) □

Proposition 3. *Let $K = K_2 = [-1,1] \times \mathbb{R}$ and suppose $f \in \mathbb{R}[x,y]$. The following are equivalent:*

(i) $f \gg 0$ on K;

(ii) $f > 0$ on K and $f^*(0,1,0) > 0$;

(iii) $f > 0$ on K and the leading term of f as a polynomial in y is of the form cy^d, where $c \in \mathbb{R}$ and $d = \deg f$.

Proof. It is not too hard to see that $\overline{K^*}$ consists of the intersection of the unit sphere with the set of (u,v,w) satisfying $|u| \le w$ and $w \ge 0$. Then (i) $\Rightarrow$ (ii) is clear since $(0,1,0) \in \overline{K^*}$. Suppose that (ii) holds. Let $d = \deg f$ and write $f = F_0 + \cdots + F_d$, where F_i is the homogeneous part of f of degree i, so that $f^*(x,y,z) = \sum_{j=0}^{d} z^{d-j} F_j(x,y)$. Then $f^*(0,1,0) = F_d(0,1)$, which implies $F_d(0,1) > 0$. Hence $F_d(x,y)$ must be of the form cy^d.

Finally, suppose that (iii) holds. We need to show that $f(u,v,w) > 0$ for $(u,v,w) \in S^2$ with $|u| < w$. If $w = 0$, then $u = 0$ and $(u,v,w) = (0,1,0)$; recall that $f(0,1,0) > 0$ by hypothesis. If $w > 0$, then (u,v,w) is in K^*, hence $(u/w, v/w) \in K$. Since $f(u/w, v/w) > 0$, we have $f^*(u,v,w) > 0$.

□

Our final result is that $(*)$ holds for K_2. The proof uses an idea from [9]: For $g(x,y) \gg 0$ on K_2, fix $y = a$ and look at the one variable polynomial $g(x,a)$. This is positive on $[-1,1]$, a compact set, so we have representations of each $g(x,a)$ in $T_{1\pm x} \subseteq \mathbb{R}[x]$. We "glue" these together to form a representation of $g(x,y)$ in T_2.

As in [10], for $f(x) \in \mathbb{R}[x]$ of degree d, we define $\tilde{f}(x)$, the *Goursat transform* of f, by the equation

$$\tilde{f}(x) = (1+x)^d f\left(\frac{1-x}{1+x}\right).$$

We collect some easy results from [10] about the Goursat transform:

Lemma 1. *If $f(x) \in \mathbb{R}[x]$ of degree d, then*

1. $\deg \tilde{f} \le d$ *with equality iff* $f(-1) \ne 0$*;*
2. $\tilde{\tilde{f}} = 2^d f$*;*
3. $f > 0$ *on* $[-1,1]$ *iff* $\tilde{f} > 0$ *on* $[0,\infty)$ *and* $\deg(\tilde{f}) = d$*.*

We also need a quantitative version of an old result, proved as [10, Theorem 6]. This is stated using the improved bound for Pólya's Theorem from [11].

Proposition 4. *Suppose $f(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$ and*

$$\lambda = \min\{f(x) \mid -1 \le x \le 1\} > 0.$$

Let $\tilde{f}(x) = \sum_{i=0}^{d} a_i (1-x)^i (1+x)^{d-i} = \sum_{i=0}^{d} b_i x^i$ and let

$$\tilde{L}(f) := \max\{|b_i| \mid i = 0, \ldots d\}.$$

Finally, let

$$N(f) := \frac{d(d-1)}{2} \frac{\tilde{L}(f)}{\lambda}.$$

If $N > N(f)$, then the coefficients of the polynomial $(1+x)^N \tilde{f}(x)$ are positive.

Theorem 3. *Given $N, d \in \mathbb{N}$, there exist polynomials $C_i \in \mathbb{R}[x_0, \ldots, x_d]$, $0 \leq i \leq N + d$, with the following property: If $f(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$ is positive on $[-1, 1]$ and $N > N(f)$, then $C_i(a_0, \ldots, a_d) > 0$ and*

$$f(x) = \sum_{i=0}^{N+d} C_i(a_0, \ldots, a_d)(1 + x)^i (1 - x)^{N+d-i}.$$

Proof. Write

$$(1 + x)^N \tilde{f} = \sum_{j=0}^{N+d} b_j x^j, \tag{6}$$

where $b_j > 0$ for all j.

For $0 \leq j \leq N + d$, let $c_j(t_0, \ldots, t_d)$ be the coefficient of x^j in the expansion of

$$(1 + x)^N \sum_{j=0}^{d} t_j (1 - x)^i (1 + x)^{d-i};$$

clearly each $c_j \in \mathbb{R}[t_0, \ldots, t_d]$, and by construction, $b_j = c_j(a_0, \ldots, a_d)$.

Now apply the Goursat transformation to both sides of (6) to obtain

$$2^{N+d} f = \sum_{j=0}^{N+d} b_j (1 - x)^j (1 + x)^{N+d-j}.$$

Setting $C_j = 2^{-(N+d)} c_j$, we have that $C_j(a_1, \ldots, a_d) > 0$ for all j and $f = \sum C_j(a_0, \ldots, a_d) x^j$. □

Example 1. For linear polynomials the proposition is easy. Suppose $f(x) = a_1 x + a_0 > 0$ on $[-1, 1]$, then we can find a representation of the form specified with $N = 0$. In this case, we have

$$f(x) = C_0(a_0, a_1) \cdot (1 - x) + C_1(a_0, a_1) \cdot (1 - x),$$

with $C_0(t_0, t_1) = \frac{1}{2} t_0 - \frac{1}{2} t_1$ and $C_1(t_0, t_1) = \frac{1}{2} t_0 + \frac{1}{2} t_1$. Note that $f(x) > 0$ on $[-1, 1]$ implies immediately that $C_j(a_0, a_1) > 0$.

Suppose we are given $g \gg 0$ on K_2. For each $r \in \mathbb{R}$, define $g_r(x) \in \mathbb{R}[x]$ by $g_r(x) = g(x, r)$ and note that $g_r(x) > 0$ on $[-1, 1]$ for all r. Let L_r denote $\tilde{L}(g_r)$ and let $\lambda_r = \min\{g_r(x) \mid -1 \leq x \leq 1\}$.

Lemma 2. *Suppose $g \gg 0$ on K_2. Then there is $u > 0$ such that*

$$\frac{\tilde{L}_r}{\lambda_r} \leq u$$

for all r.

Proof. This is similar to [9, Prop. 1]. Let $d = \deg_x g$ and $m = \deg_y g$, and write

$$g(x,y) = \sum_{i=0}^{m} h_i(x) y^i.$$

Since $g \gg 0$ on K_2, by Proposition 3, the leading term in g as a polynomial in y, $h_m(x)$, is actually a positive real constant c. For each i, $0 \leq i \leq m-1$, there is $M_i > 0$ such that $h_i(x) < M_i$ for $x \in [-1,1]$. Then, on $[-1,1]$,

$$g_r(x) \geq cr^m - \sum_{j=0}^{m-1} M_j r^j > wr^m$$

for some positive constant w and $|r|$ sufficiently large. In other words, for sufficiently large $|r|$, we have $\lambda_r \geq wr^m$.

Now write $g(x,y)$ as a polynomial in x: $g = \sum_{i=0}^{d} k_i(y) x^i$. Then $\deg k_i(y) \leq m$ for all i, by assumption. This means that the coefficients of $g_r(x)$ are $\mathcal{O}(|r|^m)$ as $|r| \to \infty$. The coefficients of $\tilde{g}_r(x)$ are linear combinations of the coefficients of $g_r(x)$, so the same is true for $\tilde{g}_r(x)$. From this is follows that

$$\frac{\tilde{L}_r}{\lambda_r} \leq \frac{w' r^m}{wr^m}$$

for some constant w' and $|r|$ sufficiently large and the result is clear.

□

Theorem 4. (∗) *holds for* K_2*: If* $g \gg 0$ *on* K_2*, then* $g \in T_2$*.*

Proof. Let u be as in the lemma and set $N = \frac{d(d-1)}{2} u$, so that we can apply Proposition 3 to each g_r with this N.

For $i = 0, \ldots, N+d$, let $C_j \in \mathbb{R}[t_0, \ldots, t_d]$ be as in the proposition. Writing $g(x,y) = \sum_{i=0}^{d} e_i(y) x^i$, define $P_1, \ldots, P_{d+N} \in \mathbb{R}[y]$ by $P_j = C_j(e_0(y), \ldots, e_d(y))$. Then the conclusion of Theorem 3 implies that

$$g(x,y) = \sum_{j=0}^{d+N} P_j(y)(1-x)^i(1+x)^{N+d-i}. \tag{7}$$

For each $r \in \mathbb{R}$ and each j, we have $P_j(r) = C_j(e_0(r), \ldots, e_d(r))$ and then, since $\{e_0(r), \ldots, e_d(r)\}$ are the coefficients of g_r, it follows from the conclusion of Proposition 3 that $P_j(r) > 0$; that is $P_j > 0$ on $\mathbb{R}$ for all j. Thus, each $P_j(y)$ is a sum of two squares of polynomials and, plugging sos representations of the P_j's into (7) yields a representation of g in T_2. □

Example 2. Let $g(x,y) = y^2 - xy + y + 1$, then for each $r \in \mathbb{R}$,

$$g_r(x) = -rx + (r^2 + r + 1) > 0$$

on $[-1,1]$. By the above, we have, for each r, the representation

$$g_r = \frac{1}{2}(r^2+2r+1)(1-x) + \frac{1}{2}(r^2+1)(1+x)$$

Then $C_0(y) = y^2+2y+1 = (y+1)^2$ and $C_1(y) = y^2+1$ yields the representation

$$g(x,y) = \frac{1}{2}(y+1)^2(1-x) + \frac{1}{2}(y^2+1)(1+x) \in T_2$$

References

[1] M. Choi, T. Y. Lam, and B. Reznick, *Sums of squares of real polynomials*, Proc. Sym. Pure Math. **58.2** (1995), 103–126.

[2] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formenquadraten*, Math. Ann. **32** (1888), 342-350; see Ges. Abh. **2**, 154-161, Springer, Berlin, 1935, reprinted by Chelsea, New York 1981.

[3] S. Kuhlmann and M. Marshall, *Positivity, sums of squares, and the multidimensional moment problem*, Trans. Amer. Math Soc. **354** (2002), 4285-4301.

[4] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim. **11** (2001), 796-817.

[5] J.B. Lasserre, *SDP versus LP relaxations for polynomial programming*, Math. Oper. Res. **27** (2002), 347-360.

[6] M. Marshall, *Optimization of polynomial functions*, Canad. Math. Bull.**46** (2003) 575-587.

[7] T. Motzkin, *The arithmetic-geometric inequalities*, in Inequalities, O. Shisha, ed., Proc. Symp. Wright-Patterson AFB, August 19-27, 1965, 205-224. Academic Press 1967.

[8] G. Pólya and G. Szegö, *Problems and Theorems in Analysis II*, Springer-Verlag, New York, et al., 1976.

[9] V. Powers, *Positive polynomials and the moment problem for cylinders with compact cross-section*, J. Pure and Appl. Alg. **188** (2004), 217-226.

[10] V. Powers and B. Reznick, *Polynomials that are positive on a interval*, Trans. Amer. Math. Soc. **352** (2000), 4677-4692.

[11] V. Powers and B. Reznick, *A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra*, J. Pure and Applied Alg. **164** (2001), 221-229.

[12] M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Ind. Univ. Math. J. (1993), 969-984.

[13] B. Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc. **63**, 1992.

[14] R. M. Robinson, *Some definite polynomials which are not sums of squares of real polynomials*, Selected questions of algebra and logic (a collection dedicated to the memory of A. I. Mal'cev), Isdat. "Nauka" Sibirsk. Otdel. Novosibirsk (1973), 264-282, Abstracts in Notices Amer. Math. Soc. 16 (1969), p. 554.

[15] C. Scheiderer, *Sums of squares of regular functions in real algebraic varieties*, Trans. Amer. Math. Soc. **352** (2000), 1039–1069.

[16] K. Schmüdgen, *The K-moment problem for compact semialgebraic sets*, Math. Ann. **289** (1991), 203–206.

[17] G. Stengle, *Complexity estimates for the Schm̈udgen Positivstellensatz*, J. Complexity **12** (1996), 167-174.