AN ALGORITHM FOR SUMS OF SQUARES OF REAL POLYNOMIALS

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INTRODUCTION

We present an algorithm to determine if a real polynomial is a sum of squares (of polynomials), and to find an explicit representation if it is a sum of squares. This algorithm uses the fact that a sum of squares representation of a real polynomial corresponds to a real, symmetric, positive semi-definite matrix whose entries satisfy certain linear equations.

SUMS OF SQUARES AND GRAM MATRICES

We fix $n$ and use the following notation in $R := \mathbb{R}[x_1, \ldots, x_n]$: For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, let $x^\alpha$ denote $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. For $m \in \mathbb{N}_0$, set $\Lambda_m := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \mid \alpha_1 + \cdots + \alpha_n \leq m\}$. Then $f \in R$ of degree $m$ can be written $f = \sum_{\alpha \in \Lambda_m} a_\alpha x^\alpha$. We say $f$ is sos if $f$ is a sum of squares of elements in $R$.

Suppose $f$ is sos, say $f$ is a sum of $t$ squares in $R$, then $f$ must have even degree, say $2m$. Thus $f = \sum_{i=1}^t h_i^2$, where each $h_i$ has degree $\leq m$. Suppose $|\Lambda_m| = k$, then we order the elements of $\Lambda_m$ in some way: $\Lambda_m = \{\beta_1, \ldots, \beta_k\}$. Set $\bar{x} := (x^{\beta_1}, \ldots, x^{\beta_k})$ and let $A$ be the $k \times t$ matrix with $i$th column the coefficients of $h_i$. Then the equation $f = \sum h_i^2$ can be written $f = \bar{x} \cdot (AA^T) \cdot \bar{x}^T$.

The symmetric $k \times k$ matrix $B := AA^T$ is sometimes called a Gram matrix of $f$ (associated to the $h_i$’s). Note that $B$ is psd (= “positive semi-definite”), i.e., $\bar{y} \cdot B \cdot \bar{y}^T \geq 0$ for all $\bar{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$.

The following theorem, in a different form, can be found in [CLR]. However we include the theorem and its proof for the convenience of the reader.

**Theorem 1.** Suppose $f \in R$ is of degree $2m$ and $\bar{x}$ is as above. Then $f$ is a sum of squares in $R$ iff there exists a real, symmetric, psd matrix $B$ such that $f = \bar{x} \cdot B \cdot \bar{x}^T$.

Given such a matrix $B$ of rank $t$, then we can construct polynomials $h_1, \ldots, h_t$ such that $f = \sum h_i^2$ and $B$ is a Gram matrix of $f$ associated to the $h_i$’s.

**Proof.** If $f = \sum h_i^2$ is sos, then as above we take $B = A \cdot A^T$, where $A$ is the matrix whose columns are the coefficients of the $h_i$’s.
Suppose there exists a real, symmetric, psd matrix $B$ such that $f = \bar{x} \cdot B \cdot \bar{x}^T$ and rank $B = t$. Since $B$ is real symmetric of rank $t$, there exists a real matrix $V$ and a real diagonal matrix $D = \text{diag}(d_1, \ldots, d_t, 0, \ldots, 0)$ such that $B = V \cdot D \cdot V^T$ and $d_i \neq 0$ for all $i$. Since $B$ is psd we have $d_i > 0$ for all $i$. Then

\[(*) \quad f = \bar{x} \cdot V \cdot D \cdot V^T \cdot \bar{x}^T.\]

Suppose $V = (v_{i,j})$, then for $i = 1, \ldots, t$, set $h_i := \sqrt{d_i} \sum_{j=1}^{k} v_{j,i} x^{\beta_i} \in R$. It follows from $(*)$ that $f = h_1^2 + \cdots + h_t^2$. \hfill \square

Thus to find a representation of $f$ as a sum of squares, we need only find a matrix $B$ which satisfies the theorem. Further, if we can show that no such $B$ exists, then we know that $f$ is not a sum of squares in $R$. Note that if $f = \sum a_\alpha x^\alpha$ and $B = (b_{i,j})$ is a $k \times k$ symmetric matrix then by “term inspection”, $f = \bar{x} \cdot B \cdot \bar{x}^T$ iff for all $\alpha \in \Lambda_{2m}$,

\[(**) \quad \sum_{\beta_i + \beta_j = \alpha} b_{i,j} = a_\alpha.\]

**The algorithm**

Given $f \in R$ of degree $2m$.

1. Let $B = (b_{i,j})$ be a symmetric matrix with variable entries. Solve the linear system that arises from $f = \bar{x} \cdot B \cdot \bar{x}^T$, i.e., solve the linear system defined by equations of the form $(**)$ above, with one equation for each $\alpha \in \Lambda_{2m}$. Note that each variable $b_{i,j}$ appears in only one equation, hence the solution is found by setting all but one variable in each row equal to a parameter and solving for the remaining variable. Then the solution is given by $B = B_0 + \lambda_1 B_1 + \cdots + \lambda_l B_l$, where each $B_i$ is a real symmetric $k \times k$ matrix and $\lambda_1, \ldots, \lambda_l$ are the parameters. In this case $l = k(k+1)/2 - |\Lambda_{2m}|$.

**Remark.** In general, the size of the matrix $B$ grows rapidly as the number of variables and the degree of the polynomial increases, since $k = |\Lambda_m| = \binom{n+m}{n}$.

However for a particular polynomial we can sometimes decrease the size of the Gram matrix by eliminating unnecessary elements of $\Lambda_m$. For example, suppose $\alpha \in \Lambda_{2m}$, $\alpha = 2\beta$, and $\alpha$ cannot be written in any other way as a sum of elements in $\Lambda_m$. Then if the coefficient of $\alpha$ in $f$ is 0, we know $x^\beta$ cannot occur in any $h_i$, cf. [CL, §2] and [CLR, 3.7].

2. We want to find values for the $\lambda_i$’s that make $B = B_0 + \lambda_1 B_1 + \cdots + \lambda_l B_l$ psd. As is well known, $B$ is psd iff all eigenvalues are non-negative. Let $F(y) = y^k + b_{k-1} y^{k-1} + \cdots + b_0$ be the characteristic polynomial of $B$. Note that each $b_i \in \mathbb{R}[\lambda_1, \ldots, \lambda_l]$. By Descartes’ rule of signs, which is exact for a polynomial with only real roots, $F(y)$ has only non-negative roots iff $(-1)^{(i+k)} b_i \geq 0$ for all $i = 0, \ldots, k-1$. Hence we consider the semi-algebraic set

\[S := \{(\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l \mid (-1)^{(i+k)} b_i (\lambda_1, \ldots, \lambda_l) \geq 0\}.\]
Then $f$ is sos iff $S$ is nonempty, and a point in $S$ corresponds to a matrix satisfying the conditions of Theorem 1.

**Remark.** There are several different algorithms for determining whether or not a semi-algebraic set is empty, for example using quantifier elimination. Unfortunately, none of these algorithms are practical apart from “small” examples. For more on this topic, see e.g. [BCR], [C], [GV], [R].

3. Given a matrix $B = (b_{i,j})$ which satisfies the conditions of Theorem 1, then we use the procedure in the proof of the theorem to find a representation of $f$ as a sum of squares.

**Example 1.** Let $f = x^2y^2 + x^2 + y^2 + 1$, then $f$ is visibly a sum of squares. We want to find all possible representations of $f$ as a sum of squares. Note that by the remark above, if $f = \sum h_i^2$ then the only monomials that can occur in the $h_i$’s are $xy, x, y, 1$. So set $\beta_1 = (1, 1), \beta_2 = (1, 0), \beta_3 = (0, 1)$, and $\beta_4 = (0, 0)$. Then the linear system in step 1 of the algorithm is

\[
\begin{align*}
b_{1,1} &= 1, \quad 2b_{1,2} = 0, \quad 2b_{1,3} = 0, \quad 2b_{1,4} + 2b_{2,3} = 0 \\
b_{2,2} &= 1, \quad 2b_{2,4} = 0 \\
b_{3,3} &= 1, \quad 2b_{3,4} = 0 \\
b_{4,4} &= 1
\end{align*}
\]

Thus the general form of a Gram matrix for $f$ is

\[
B = \begin{bmatrix}
1 & 0 & 0 & \lambda \\
0 & 1 & -\lambda & 0 \\
0 & -\lambda & 1 & 0 \\
\lambda & 0 & 0 & 1
\end{bmatrix}.
\]

The characteristic polynomial of $B$ is

\[y^4 - 4y^3 + (6 - 2\lambda^2)y^2 + (4\lambda^2 - 4)y + (\lambda^4 - 2\lambda^2 + 1),\]

thus $B$ is psd iff $-1 \leq \lambda \leq 1$. Note that rank $B = 2$ if $\lambda = \pm 1$, otherwise rank $B = 4$. Hence $f$ can be written as a sum of 2 or 4 squares.

We have $B = V \cdot D \cdot V^T$, where $D = \text{diag}(1,1,1 - \lambda^2,1 - \lambda^2)$ and $V = \begin{bmatrix}1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -\lambda & 1 & 0 \\
\lambda & 0 & 0 & 1\end{bmatrix}$. This yields

\[f = (xy + \lambda)^2 + (x - \lambda y)^2 + (\sqrt{1 - \lambda^2}y)^2 + (\sqrt{1 - \lambda^2})^2.\]

Note that $\lambda = 0$ yields the original representation of $f$ as a sum of 4 squares.
Example 2. Let \( f(x, y, z) = x^4 + 2x^2y^2 + x^3z + z^4 \). A Gram matrix for \( f \) would be of the form
\[
\begin{bmatrix}
1 & 0 & 2 & \lambda \\
0 & 2 & 0 & 0 \\
2 & 0 & -2\lambda & 0 \\
\lambda & 0 & 0 & 1
\end{bmatrix}.
\]
In this case, \( S \subseteq \{-8 - 4\lambda + 4\lambda^2 \geq 0, -8 - 4\lambda \geq 0\} = \emptyset \). Hence \( f \) is not sos.

Example 3. Let \( f(x, y, z) = x^6 + 4x^3y^2z + y^6 + 2y^4z^2 + y^2z^4 + 4z^6 \). In this case the only exponents that can occur in the \( h_i \)'s are \( \{(3, 0, 0), (0, 3, 0), (0, 2, 1), (0, 1, 2), (0, 0, 3)\} \). We get
\[
B = \begin{bmatrix}
1 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & r & s \\
2 & 0 & 2 - 2r & -s & t \\
0 & r & -s & 1 - 2t & 0 \\
0 & s & t & 0 & 4
\end{bmatrix}
\]
as the general form of a Gram matrix.

The corresponding semi-algebraic set is \( S = \{-2r - 2t + 9 \geq 0, -r^2 + 4rt - 14r - 2s^2 - t^2 - 16t + 25 \geq 0, 2r^3 - 7r^2 + 2rs^2 + 24rt - 30r + 2s^2t - 10s^2 + 2t^3 - 3t^2 - 34t + 19 \geq 0, 10r^3 + 2r^2t - 10r^2 - 2rs^2t + 4rs^2 + 36rt - 26r + s^4 + 6s^2t - 10s^2 + 4t^3 - 3t^2 - 4t - 6 \geq 0, 8r^3 + r^2t^2 + 8r^2 + 2rs^2t + 2r s^2 + 16rt - 8r + s^4 - 4s^2t - 2s^2 + 2t^3 - t^2 - 16t - 8 \geq 0\} \).

If we set \( s = t = 0 \), we see \( (-1, 0, 0) \in S \), and setting \( s = 0 \) and \( r = -2 \) we see \( (-2, 0, -3/2) \in S \). In particular, \( S \) is nonempty and so \( f \) is a sum of squares.

Using \( (-1, 0, 0) \),
\[
B = \begin{bmatrix}
1 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
2 & 0 & 4 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}.
\]

Note that rank \( B = 3 \), so this gives \( f \) as a sum of 3 squares. In this case we get
\[
f = (x^3 + 2y^2 z)^2 + (y^3 - yz^2)^2 + (2z^3)^2.
\]

Using \( (-2, 0, -3/2) \),
\[
B = \begin{bmatrix}
1 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & -2 & 0 \\
2 & 0 & 6 & 0 & -3/2 \\
0 & -2 & 0 & 4 & 0 \\
0 & 0 & -3/2 & 0 & 4
\end{bmatrix}.
\]

Note rank \( B = 4 \). Proceeding as before we get
\[
f = (x^3 + 2y^2 z)^2 + (y^3 - 2yz^2)^2 + (\sqrt{2}y^2 z - 3\sqrt{2}/4z^3)^2 + (\sqrt{23}/8z^3)^2.
\]

Remark. Let \( (K, \leq) \) be any ordered field with real closure \( R \), and suppose \( f \in K[x_1, \ldots, x_n] \). Then we can easily extend the algorithm to decide whether or not \( f \) is a sum of squares in \( R[x_1, \ldots, x_n] \).
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REFERENCES


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