A New Proof of Hilbert's Theorem on Ternary Quartics

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Abstract

Hilbert proved that a non-negative real quartic form $f(x, y, z)$ is the sum of three squares of quadratic forms. We give a new proof which shows that if the plane curve Q defined by f is smooth, then f has exactly 8 such representations, up to equivalence. They correspond to those real 2-torsion points of the Jacobian of Q which are not represented by a conjugation-invariant divisor on Q.

1. Introduction

A *ternary quartic* is a homogeneous polynomial $f(x, y, z)$ of degree 4 in three variables. If f has real coefficients, then f is *non-negative* if $f(x, y, z) \geq 0$ for all real x, y, z. Hilbert [5] showed that every non-negative real ternary quartic form is a sum of three squares of quadratic forms. His proof (see [8], [9] for modern expositions) was non-constructive: The map

$$\pi\colon\ (p, q, r) \ \longmapsto\ p^2 + q^2 + r^2$$

from triples of real quadratic forms to non-negative quartic forms is surjective, as it is both open and closed when restricted to the preimage of the (dense) connected set of non-negative quartic forms which define a smooth complex plane curve. An elementary and constructive approach to Hilbert's theorem was recently begun by Pfister [6].

A *quadratic representation* of a complex ternary quartic form $f = f(x, y, z)$ is an expression

$$f \ = \ p^2 + q^2 + r^2 \tag{1}$$

where p, q, r are complex quadratic forms. A representation $f = (p')^2 + (q')^2 + (r')^2$ is *equivalent* to this if p, q, r and p', q', r' have the same linear span in the space of quadratic forms.

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Powers and Reznick [7] investigated quadratic representations computationally, using the Gram matrix method of [1]. In several examples of non-negative ternary quartics, they always found 63 inequivalent representations as a sum of three squares of complex quadratic forms; 15 of these were sums or differences of squares of real forms. We explain these numbers, in particular the number 15, and show that precisely 8 of the 15 are sums of squares.

If the complex plane curve Q defined by $f = 0$ is smooth, it has genus 3, and so the Jacobian J of Q has $2^6 - 1 = 63$ non-zero 2-torsion points. Coble [2] showed that these are in one-to-one correspondence with equivalence classes of quadratic representations of f. If f is real, then Q and J are defined over $\mathbb{R}$. The non-zero 2-torsion points of $J(\mathbb{R})$ correspond to *signed quadratic representations* $f = \pm p_1 \pm p_2 \pm p_3$, where p_i are real quadratic forms. If f is also non-negative, the real Lie group $J(\mathbb{R})$ has two connected components, and hence has $2^4 - 1 = 15$ non-zero 2-torsion points. We use Galois cohomology to determine which 2-torsion points give rise to sum of squares representations over $\mathbb{R}$.

Theorem 1 *Suppose that $f(x, y, z)$ is a non-negative real quartic form which defines a smooth complex plane curve Q. Then the inequivalent representations of f as a sum of three squares are in one-to-one correspondence with the eight 2-torsion points in the non-identity component of $J(\mathbb{R})$, where J is the Jacobian of Q.*

Wall [10] studies quadratic representations of (possibly singular) complex ternary quartic forms f. Again, in the irreducible case, the non-trivial 2-torsion points on the generalized Jacobian give equivalence classes of quadratic representations of f. These representations are special in that they have no basepoints.

Quadratic representations with a given base locus B correspond to the 2-torsion points on the Jacobian of a curve $\widetilde{Q}$, which is the image of Q under the complete linear series of quadrics through B. Classifying all possibilities for B gives the number of inequivalent quadratic representations of f. If f is real and non-negative, this classification, together with arguments from Galois cohomology, gives all inequivalent representations of f as a sum of squares. This complete analysis will appear in an unabridged version.

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2. Basepoint-free quadratic representations

Let $f(x, y, z)$ be an irreducible quartic form over $\mathbb{C}$, and let Q be the complex plane curve $f = 0$. The Picard group $\operatorname{Pic}(Q)$ of Q is the group of Weil divisors on the regular part of Q, modulo divisors of rational functions which are invertible around the singular locus of Q. Let J_Q be the generalized Jacobian of Q, so that $J_Q(\mathbb{C})$ is the identity component of $\operatorname{Pic}(Q)$. Its structure may be determined from the Jacobian of the normalization $\widetilde{Q}$ of Q via the exact sequence [4, Ex. II.6.9]

$$0 \longrightarrow \bigoplus_{p \in Q} \widetilde{\mathcal{O}}_p^* / \mathcal{O}_p^* \longrightarrow J_Q(\mathbb{C}) \longrightarrow J_{\widetilde{Q}}(\mathbb{C}) \longrightarrow 0\,,$$

where $\mathcal{O}_p$ is the local ring of Q at p, $\widetilde{\mathcal{O}}_p$ is its normalization, and * indicates the group of units.

The *base locus* B of a quadratic representation (1) of f is the zero scheme of the homogeneous ideal generated by the forms p, q, r. The closed subscheme B is supported on the singular locus of Q. We say that (1) is *basepoint-free* if B is empty.

Proposition 1 (Coble [2], Wall [10]) *The non-trivial 2-torsion points of J_Q are in one-to-one correspondence with the equivalence classes of basepoint-free quadratic representations of f.*

Proof. Given a basepoint-free quadratic representation (1), consider the map

$$\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2, \quad x \mapsto \big(p(x) : q(x) : r(x)\big).$$

The image of Q under φ is the conic C defined by the equation $y_0^2 + y_1^2 + y_2^2 = 0$. Let y be a point in C whose preimages are regular points of Q. Then $\varphi^*(y)$ is an effective divisor of degree 4 that is not the divisor of a linear form. Indeed, after a linear change of coordinates we can assume $y = (0 : 1 : i)$. A linear form vanishing on $\varphi^*(y)$ would divide each conic $\alpha p + \beta(q + ir)$ through $\varphi^*(y)$, and thus would divide

$$f \;=\; p^2 + (q + ir)(q - ir)\,,$$

contradicting the irreducibility of f.

Fix a linear form ℓ that does not vanish at any singular point of Q. Then $L := \mathrm{div}(\ell)$ is an effective divisor of degree 4 on Q. Let $\zeta = [\varphi^*(y) - L]$. Since $2y$ is the divisor of a linear form (the tangent line to C at y), $\varphi^*(2y)$ is the divisor on Q of a quadratic form. Thus $2\zeta = 0$. Moreover, $\zeta \neq 0$ as $\varphi^*(y)$ is not the divisor of a linear form. The 2-torsion point ζ of J_Q depends only upon the map φ.

Conversely, suppose that $\zeta \in J_Q(\mathbb{C})$ is a non-zero 2-torsion point. Let $D \neq D'$ be effective divisors which represent the class $\zeta + [L]$ in $\mathrm{Pic}(Q)$. As Q has arithmetic genus 3, the Riemann-Roch Theorem implies that there is a pencil of such divisors. Then $2D$, $2D'$ and $D + D'$ are effective divisors of degree 8, and are all linearly equivalent to $2L$, the divisor of a conic. By the Riemann-Roch Theorem there are quadratic forms q_0, q_1 and q_2 such that

$$\mathrm{div}(q_0) = 2D, \quad \mathrm{div}(q_1) = 2D' \quad \text{and} \quad \mathrm{div}(q_2) = D + D'.$$

Therefore, the rational function $g := q_0 q_1 / q_2^2$ on Q is constant. Scaling q_1 and q_2 appropriately, we may assume that $g \equiv 1$ on Q and also that $f = q_0 q_1 - q_2^2$. Diagonalizing the quadratic form $q_0 q_1 - q_2^2$ gives a quadratic representation for f. This defines the inverse of the previous map. □

3. Quadratic representations of real quartics

Suppose now that f is a non-negative real quartic form defining a real plane curve Q with complexification $Q_{\mathbb{C}} = Q \otimes_{\mathbb{R}} \mathbb{C}$. The elements of $\mathrm{Pic}(Q)$ can be identified with those divisor classes in $\mathrm{Pic}(Q_{\mathbb{C}})$ that are represented by a conjugation-invariant divisor. Let J be the generalized Jacobian of Q.

If $\zeta \in J(\mathbb{C})$ is the 2-torsion point corresponding to a signed quadratic representation

$$f \;=\; \pm p^2 \pm q^2 \pm r^2$$

consisting of real polynomials p, q, r, then $\zeta = \overline{\zeta}$, i.e., $\zeta \in J(\mathbb{R})$.

Conversely, let $0 \neq \zeta \in J(\mathbb{R})$ with $2\zeta = 0$. Choose a real linear form ℓ not vanishing on the singular points of Q, and let $L = \mathrm{div}(\ell)$. We can choose effective divisors $D \neq \overline{D}$ on $Q_{\mathbb{C}}$ representing the class $\zeta + [L]$. Then $2D$, $2\overline{D}$ and $D + \overline{D}$ are each equivalent to $2L$. Let r be a real quadratic form with divisor $D + \overline{D}$, and let g be a (complex) quadratic form with divisor $2D$ (both divisors taken on $Q_{\mathbb{C}}$).

Since $D \sim \overline{D}$, there is a rational function h on $Q_{\mathbb{C}}$, invertible around Q_{sing}, with $\mathrm{div}(h) = D - \overline{D}$. Let $c = h\overline{h}$, a nonzero real constant on Q. Since $\mathrm{div}(r) = \mathrm{div}(g) + \mathrm{div}(h)$, there is a complex number $\alpha \neq 0$ with $\frac{r}{g} = \alpha h$ on Q, which implies that

$$c|\alpha|^2 \;=\; \frac{r}{g}\,\frac{\overline{r}}{\overline{g}} \;=\; \frac{r^2}{p^2 + q^2}$$

on Q, where p, q are the real and imaginary parts of $g = p + iq$. So the quartic form

$$u \;:=\; r^2 - c\,|\alpha|^2 (p^2 + q^2)$$

vanishes identically on Q. Since $u \neq 0$, f is a constant multiple of u. If $c > 0$, we get a signed quadratic representation of f, with both signs $\pm$ occuring. If $c < 0$, f must be a positive multiple of u since f is non-negative, and we get a representation of f as a sum of three squares of real forms.

We now calculate the sign of c. For this we use the exact sequence

$$0 \to \mathrm{Pic}(Q) \to \mathrm{Pic}(Q_{\mathbb{C}})^G \xrightarrow{\partial} \mathrm{Br}(\mathbb{R}) \to H^2_{\text{ét}}(Q, \mathbb{G}_m) \tag{2}$$

of étale cohomology groups. It arises from the Hochschild-Serre spectral sequence for the Galois covering $Q_{\mathbb{C}} \to Q$ and coefficients $\mathbb{G}_m$. Here $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathrm{Pic}(Q_{\mathbb{C}})$ by conjugation, and $\mathrm{Pic}(Q_{\mathbb{C}})^G$ is the group of G-invariant divisor classes. Moreover, $\mathrm{Br}(\mathbb{R}) = H^2_{\text{ét}}(\mathrm{Spec}\,\mathbb{R}, \mathbb{G}_m)$ is the Brauer group of $\mathbb{R}$ (which is of order 2), and $\mathrm{Br}(\mathbb{R}) \to H^2_{\text{ét}}(Q, \mathbb{G}_m)$ is the restriction map.

It is easy to see that $c < 0$ if and only if $\partial(\zeta)$ is the non-trivial class in $\mathrm{Br}(\mathbb{R})$. If Q has an $\mathbb{R}$-point, then $\mathrm{Br}(\mathbb{R}) \to H^2_{\text{ét}}(Q, \mathbb{G}_m)$ has a splitting given by that point, and hence ∂ vanishes identically.

If Q is smooth, then f non-negative forces $Q(\mathbb{R}) = \emptyset$, and the map $\mathrm{Br}(\mathbb{R}) \to H^2_{\text{ét}}(Q, \mathbb{G}_m)$ is zero. In this case, $Pic(Q_C)^G$ contains an odd degree divisor if and only if the genus of Q is even and $J(\mathbb{R})^0$, the identity connected component of the real Lie group $J(\mathbb{R})$, is the kernel of the restriction $J(\mathbb{R}) \to \mathrm{Br}(\mathbb{R})$ of ∂ [11,3]. Since in our case $g(Q) = 3$, this implies that the sequence

$$0 \to J(\mathbb{R})^0 \to J(\mathbb{R}) \xrightarrow{\partial} \mathrm{Br}(\mathbb{R}) \to 0$$

is (split) exact. If Q is singular with $Q(\mathbb{R}) = \emptyset$, one compares sequence (2) for Q to the same sequence for the normalization $\widetilde{Q}$ of Q and concludes that $\partial\colon J(\mathbb{R}) \to \mathrm{Br}(\mathbb{R})$ is surjective as well.

We complete the proof of Theorem 1. Since f is non-negative and Q is smooth of genus 3, $J(\mathbb{R})^0 \cong (S^1)^3$ as a real Lie group. By the facts just mentioned, there exist $2^4-1 = 15$ non-zero 2-torsion elements in $J(\mathbb{R})$. The 8 that do not lie in $J(\mathbb{R})^0$, or equivalently, which cannot be represented by a conjugation-invariant divisor on $Q_{\mathbb{C}}$, are precisely those that give rise to sums of squares representations of f.

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