The Best Way to Choose a Winner

Vicki Powers
Emory University, Atlanta, GA
currently: National Science Foundation,
Division of Mathematical Sciences

Universität Konstanz,
April 14, 2014
This talk is about **Social Choice Theory**.

“Social choice” = group decisions

In Social Choice Theory we study mathematical models of situations where a group of people choose a “winner” based on the preferences of the individuals in the group.

For example....

- Elections
- Picking a winner in sports with judging or ranking: figure skating, Dancing with the Stars...
- Ranking sports team
- A group of friends deciding on a restaurant or movie.
In this talk we will...

- Introduce some mathematical models of voting/social choice.
- Discuss Arrow’s Impossibility Theorem
- Discuss recent work of D. Saari and T. McIntee relating pairwise and positional election outcomes
- Discuss new work, joint with M. Castle, generalizing the work of Saari and McIntee.

We start with two examples.
1998 Minnesota governor’s race

- Three candidates: Norm Coleman (R), Skip Humphrey (D), Jesse “the body” Ventura (I). Ventura was a pro wrestler.
- Results were Ventura 37%, Coleman 35%, Humphrey 28%, so Ventura won.
- In two candidate contests Ventura would have lost to both Coleman and Humphrey.
- The candidate liked least by more than half of the voters won the election!
1971 AP preseason college (American) football poll

- 50 sports journalists ranked 20 teams
- Points were awarded for a first-place ranking, second-place ranking, etc. The team with the most points wins.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Team</th>
<th>Points</th>
<th># of first-place rankings</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Notre Dame</td>
<td>885</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>Nebraska</td>
<td>870</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>Northwestern</td>
<td>58</td>
<td>1</td>
</tr>
</tbody>
</table>

Nebraska received a majority of first-place rankings, but was not ranked first!
Questions for the audience:

- In the two examples, do you think that the group make the “right” decision?

- Do the outcomes above reflect the “will of the people”? 

In this talk, I hope to convince you that mathematics plays a central role in understanding if the “will of the people” is reflected by the voting systems we use.
Examples of voting methods

Assume that each voter ranks the candidates from first to last.

- **Plurality**: The candidate with the most first-place rankings wins. Used in many elections. (MN governor’s race; used for half the seats in the Bundestag)

- **Borda count**: Candidates get $k - 1$ points for a first place ranking, $k - 2$ points for a second place ranking, etc. The candidate(s) with the most points wins. Used frequently for sports-related polls. (AP football poll)

- **Scoring rules**: Generalization of Borda count: use a scoring vector $(\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. $\alpha_1$ points for 1st place, etc.
Examples of voting rules

- **Instant Runoff** aka the **Hare method**: The candidate(s) with the least first-place rankings is removed, repeat this process; the candidate(s) deleted last is the winner. Used for elections in Ireland, Australia, Malta...

- **Antiplurality** The candidate with the least number of last-place votes is the winner.

- **Multistage Antiplurality** The candidate(s) with the most last-place votes is removed, repeat this process; the candidate(s) deleted last is the winner.
Felicia, Gerald, Helen, and Ivan are all running for the president of a club with 27 members. The preference lists of the 27 voters are as follows:

<table>
<thead>
<tr>
<th>Preference list</th>
<th>Number of voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>F G H I</td>
<td>12</td>
</tr>
<tr>
<td>G H I F</td>
<td>7</td>
</tr>
<tr>
<td>H I F G</td>
<td>5</td>
</tr>
<tr>
<td>I H G F</td>
<td>3</td>
</tr>
</tbody>
</table>

Under Plurality, Felicia wins. Under Borda Count, Gerald wins. Under Instant Runoff, Helen wins. There is even a reasonable-sounding voting method under which Ivan wins.

The outcome of the election depends on the voting method used!!!
The Social Choice model

- a finite set of voters \( N = \{1, 2, \ldots, n\} \).
- a finite set of **candidates** or **alternatives**, \( X = \{x_1, \ldots, x_k\} \).
- \( L(X) \) = set of linear orders of \( X \), i.e., possible rankings of the candidates. \( O(X) \) is the set of rankings with ties allowed.
- Each voter chooses a **preference list**, an element in \( L(X) \). A **profile** is a list of preference lists, one for each voter, i.e., \( R = (R_1, \ldots, R_n), R_i \in L(X) \). \( L(X)^n \) = the set of all possible profiles.

**A social welfare function** is a function \( F : L(X)^n \to O(X) \). Think of \( F(R) \) as the “social ranking” of the candidates or the outcome of the election.

**A social choice function** is a function \( F : L(X)^n \to \{ \text{non-empty subsets of } X \} \). Think of \( F(R) \) as the “winner(s)” of the election.
We’ll call social welfare/choice functions voting methods. Assume $F : L(X)^n \rightarrow L(X)(O(X))$ is a voting method.

Some obvious desirable properties of voting methods:

- A voting method is **anonymous** if all voters are treated the same, i.e., given any permutation $\pi \in S_n$ and $R \in L(X)^n$, we have $F(\pi(R)) = F(R)$.

- A voting method is **neutral** if all candidates are treated the same. Writing down the precise definition is left as an exercise to the listener. Hint: permutations are involved!
Marquis de Condorcet (1743-1794), a French philosopher, mathematician, and political scientist, wrote about voting methods and published a paper in 1785 describing what has become known as Condorcet’s Paradox.

A key idea in voting theory is **pairwise comparisons**: If $A$ and $B$ are two candidates, we write $A \succ B$ to mean that (strictly many) more voters rank $A$ ahead of $B$ than $B$ ahead of $A$. $A$ is a **Condorcet winner** if $A \succ B$ for every candidate $B \neq A$.

Condorcet noticed that it is possible for the Condorcet winner to lose under the plurality method. Remember the Minnesota governor’s election!
Condorcet proposed that the winner of an election should be the Condorcet winner, but noticed that this method does not always result in a winner. The problem, as he discovered, is that majority preferences are not transitive:

It is possible that $A \succ B$, $B \succ C$, and $C \succ A$.

A voting method satisfies the **Condorcet winner criterion**, CWC, if whenever there is a Condorcet winner, that candidate is the unique winner.

Plurality and scoring methods such as Borda count **do not** satisfy the CWC. Instant Runoff does not satisfy the CWC. Some methods that do satisfy CWC:

**Black’s Method**: Choose the CW, if one exists. Otherwise use Borda count.

**Dodgson’s Rule**: For each candidate, determine the fewest number of pairwise swaps to make them the CW. The candidate(s) with the lowest number wins.
1995 women’s world figure skating championship

- Used Best of Majority rules: results based on individual judge’s ranking of the skaters.
- With one skater left – Michelle Kwan – results were:
  1. Chen Lu (China)
  2. Nicole Bobek (U.S.)
  3. Suraya Bonaly (France)
- Kwan skated and the final results were:
  1. Chen Lu (China)
  2. Suraya Bonaly (France)
  3. Nicole Bobek (U.S.)
  4. ....
  5. Michele Kwan (U.S.)
- Even though the relative rankings of Bobek and Bonaly did not change, they switched places in the final outcome.
Independence of Irrelevant Alternatives

A voting method satisfies **Independence of Irrelevant Alternatives**, IIA, if whenever $A$ is a winner and $B$ is a loser and one or more voters change their preference list, but no voter changes their relative ranking of $A$ and $B$, then $B$ does not become a winner.

The voting method used for judging ice-skating in 1995 does not satisfy IIA. In fact, none of the voting methods described so far satisfy IIA!

Perhaps a violation of IIA doesn’t seem so bad. But imagine the following scenario...
A customer has finished their main course at a fancy restaurant.
Waiter: Do you want dessert? We have chocolate cake or crème brûlée.
Customer: I’ll have chocolate cake.
Waiter: I almost forgot – we also have apple pie tonight.
Customer: In that case, I’ll have crème brûlée.
Waiter: ?????

Individual preferences satisfy IIA (usually!!!), but voting methods do not always satisfy IIA.
Example 2

Angela (A), Bush (B), and Clinton (C) are running for president of a club with 17 members. They will use Instant Runoff. The preference lists of the club members are:

<table>
<thead>
<tr>
<th>Preference list</th>
<th># of voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B C</td>
<td>6</td>
</tr>
<tr>
<td>C A B</td>
<td>5</td>
</tr>
<tr>
<td>B C A</td>
<td>4</td>
</tr>
<tr>
<td>B A C</td>
<td>2</td>
</tr>
</tbody>
</table>

Notice that $A \succ B \succ C \succ A$ in this case. This is a cyclic profile – there is no Condorcet Winner. C has the lowest number of first place votes, so C is dropped. Then $A \succ B$ means that A wins.
There is some problem with the ballots and the club decides to revote. Meanwhile, Angela makes a great speech and the voters with preference $B A C$ decide to change to $A B C$. The profile is now:

<table>
<thead>
<tr>
<th>Preference list</th>
<th># of voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B C</td>
<td>8</td>
</tr>
<tr>
<td>C A B</td>
<td>5</td>
</tr>
<tr>
<td>B C A</td>
<td>4</td>
</tr>
</tbody>
</table>

What happens? $B$ is now dropped on the first round and $C$ wins the election.

$A$ moved up in some preference lists and went from winning to losing!
A voting method satisfies **Monotonicity** if whenever candidate $A$ is a winner and some voters move $A$ up in their rankings, then $A$ remains a winner. There are variations of this idea:

- **The No-Show Paradox**: A voter will obtain a more preferable outcome if they don’t vote.
- **The Twin Voter Paradox**: A voter with the same preference list that you have votes and this causes your favorite candidate to lose.

In 2009, a failure of (something like) the monotonicity property happened in the election for the Bundestag. The voting method used has been changed.
May’s Theorem

Suppose we have only two candidates, then we can use Majority Rule: Each voter chooses one candidate and the candidate with the most votes wins (or there is a tie if each has the same number of votes). In 1952, K. May proved:

**Theorem**

If there are only two candidates, then Majority Rule is the only voting method that is anonymous, neutral, and satisfies the monotonicity property.

If there are only two candidates, then Majority Rule is clearly the best voting method!

What if there are more than two candidates??
In 1951, Kenneth Arrow (an economist) proved his famous impossibility theorem. He received the Noble prize in Economics in 1972, mostly for this work. Here is one version:

**Arrow’s Impossibility Theorem**

*If there are more than two candidates, no social choice procedure that is neutral and anonymous satisfies both CWC and IIA.*

In other words, all voting methods have flaws!!!!
Picking the “best” voting method is a subtle, difficult challenge. What to do? We can look at questions like these:

- How likely is it that CWC, IIA, or monotonicity will fail for a given social choice procedure?
- How often does a CW fail to be a winner?
- Are there voting methods that don’t use preference lists that satisfy desirable properties?
For the rest of the talk, assume that there are three candidates, $X = \{A, B, C\}$. Even in this case, there are many open problems and questions.

Then $|L(X)| = 6$, i.e., there are 6 possible preference lists corresponding to the 6 permutations of $A, B, C$.

A profile can be given as a point $(e_1, e_2, \ldots, e_6)$ in $\mathbb{Z}_6^{\geq 0}$, where $e_i$ defined by:

<table>
<thead>
<tr>
<th>Preference</th>
<th># of voters</th>
<th>Preference</th>
<th># of voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A B C$</td>
<td>$e_1$</td>
<td>$C B A$</td>
<td>$e_4$</td>
</tr>
<tr>
<td>$A C B$</td>
<td>$e_2$</td>
<td>$B C A$</td>
<td>$e_5$</td>
</tr>
<tr>
<td>$C A B$</td>
<td>$e_3$</td>
<td>$B A C$</td>
<td>$e_6$</td>
</tr>
</tbody>
</table>

The order is important here: Notice that the fourth – $CAB$ – is the reverse of the first – $ABC$ and so on.
The subset of all profiles where $X \succ Y$ can be described by a linear inequality, e.g., $A \succ B$ for a profile $(e_1, e_2, \ldots, e_6)$ iff

$$e_1 + e_2 + e_3 > e_4 + e_5 + e_6.$$ 

“$A$ is the Plurality winner”, and similar statements, can be described by a set of linear inequalities.

For a fixed number of voters $n$, the set of all profiles that satisfy the condition is the set of lattice points (integer-valued points) in the polytope in $\mathbb{R}^6$ defined by $\sum e_i = n, e_i \geq 0$.

There are methods, and software, for counting lattice points in polytopes.
Suppose that $A \succ B$ and $A \succ C$, so that $A$ is the Condorcet Winner (CW). Intuitively, if $A$ just narrowly beats $B$ and/or $C$ in pairwise comparisons, $A$ should be more likely to lose than if $A$'s wins are of “landslide proportions”.

For candidates $X$ and $Y$, define the **pairwise tally** $P(X, Y)$ as
\[ \{ \# \text{ of voters with } X > Y \} - \{ \# \text{ of voters with } Y > X \}. \]

Clearly $X \succ Y$ iff $P(X, Y) > 0$.

Also, $P(Y, X) = -P(X, Y)$, so the set of pairwise tallies is determined by $P(A, B)$, $P(A, C)$ and $P(B, C)$. 
Example 3

Suppose there are 30 voters with the following preferences:

<table>
<thead>
<tr>
<th>Preference</th>
<th>Number of voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B C</td>
<td>11</td>
</tr>
<tr>
<td>B A C</td>
<td>7</td>
</tr>
<tr>
<td>C A B</td>
<td>5</td>
</tr>
<tr>
<td>C B A</td>
<td>7</td>
</tr>
</tbody>
</table>

The corresponding profile is (11, 0, 5, 7, 0, 7).

It’s easy to check that $P(A, B) = 2$, $P(A, C) = 6$, and $P(B, C) = 6$. Note A is the CW since $P(A, B)$ and $P(A, C)$ are both positive.
In recent work of D. Saari and T. McIntee, results are given relating the pairwise tallies to election outcomes for social choice methods such as Plurality and Borda count. For example, if the pairwise tallies are large, is it still possible for CW A to lose under Plurality?

By looking at profiles with a fixed set of pairwise tallies, we can reduce the number of variables determining a profile and make calculations and proofs more tractable.

For a fixed set of pairwise tallies, a supporting profile is any profile with these pairwise tallies.

Note that a profile \((\alpha, \beta, \gamma, \alpha, \beta, \gamma)\) has the property that \(P(X, Y) = 0\) for all \(X, Y\). This is a set of reversal pairs.
Fix a set of pairwise tallies. The **essential profile** for this set is the supporting profile with the smallest number of voters. Saari and McIntee showed that

- there is always a unique essential profile,
- there is an easy algorithm for finding it,
- the essential profile contains at most 3 non-zero entries, and
- any supporting profile can be written as the essential profile plus a set of reversal pairs.

For example, the profile \((11, 0, 5, 7, 0, 7)\) has pairwise tallies \(P(A, B) = 2, P(A, C) = 6,\) and \(P(B, C) = 6\). The essential profile is \((4, 0, 0, 0, 0, 2)\) and we have

\[
(11, 0, 5, 7, 0, 7) = (4, 0, 0, 0, 0, 2) + (7, 0, 5, 7, 0, 5).
\]
There are four types of essential profiles. If \( A \succ B \succ C \succ A \) (the cyclic case), the essential profile is of the form \((e_1, 0, e_3, 0, e_5, 0)\).

An example of the results of Saari and McIntee is the following:

\[ \text{Theorem} \]

\textit{Suppose } \( A \) \textit{is the CW and } \( n \) \textit{the number of voters. Suppose } \( P(A, B) \geq P(A, C) \), \textit{then a necessary and sufficient condition for all supporting profiles to have the CW } \( A \) \textit{as the sole Plurality winner is}

\[ n < 2P(A, C) + P(A, B). \]

For example, if \( P(A, B) = 10 \) and \( P(A, C) = 8 \), then if \( n \geq 26 \), there will always exist a profile with these pairwise tallies for which \( A \) is not the Plurality winner. To put it another way, only if \( n \leq 25 \) can you guarantee that CW \( A \) will always win under Plurality.
Saari and McIntee gave results for Plurality and Borda Count. In joint work with Mari Castle, we give similar results for Instant Runoff (IR). Our results show that with 3 candidates, IR does better than Plurality.

For example, we can show:

**Theorem**

*For fixed pairwise tallies, let $n$ be the number of voters. Suppose $P(A, B) \geq P(A, C)$, then a necessary and sufficient condition for all supporting profiles to have the CW A as the sole Instant Runoff winner is*

$$n < 2P(A, B) + P(A, C).$$

$n < 2 \times$ smaller + bigger becomes $n < 2 \times$ bigger + smaller. With $P(A, B) = 10$, $P(A, C) = 8$, then for $n \leq 27$, A will be the IR winner for all profiles.
For a fixed set of pairwise tallies, supporting profiles are determined by $\alpha, \beta, \gamma$. The number of voters, $n$, is determined by the essential profile and $\alpha, \beta, \gamma$. If we fix $n$, supporting profiles are determined by two variables.

Thus the set of profiles for which CW A is the Plurality or IR winner is the set of all lattice points in a certain polytope in the plane.

We obtain results by counting lattice points. Counting lattice points in this situation is easy using Pick’s Theorem:

**Pick’s Theorem**

Suppose $P$ is a polytope in the plane with area $U$, $i$ interior lattice points, and $b$ boundary lattice points, then

$$U = i + b/2 - 1.$$
<table>
<thead>
<tr>
<th>Preference</th>
<th>Number of voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B C</td>
<td>11</td>
</tr>
<tr>
<td>B A C</td>
<td>7</td>
</tr>
<tr>
<td>C A B</td>
<td>5</td>
</tr>
<tr>
<td>C B A</td>
<td>7</td>
</tr>
</tbody>
</table>

Recall this is \((11, 0, 5, 7, 0, 7) = (4, 0, 0, 0, 0, 2) + (7, 0, 5, 7, 0, 5)\) with \(P(A, B) = 2\), \(P(A, C) = 6\), and \(P(B, C) = 6\), as above.

\(A\) is the CW and wins under IR, but not Plurality.

If \(A\) wins under Plurality, then \(A\) wins under IR.
Profiles which have essential profile \((4, 0, 0, 0, 0, 2)\) are all profiles \((4 + \alpha, \beta, \gamma, \alpha, \beta, 2 + \gamma)\).

\(A\) has more first place rankings than \(B, C\) iff

\[
4 + \alpha + \beta > 2 + \gamma + \beta \quad (1)
\]

\[
4 + \alpha + \beta > \alpha + \gamma \quad (2)
\]

\(A\) is the Plurality winner iff both of these hold, while \(A\) is the IR winner iff at least one of these hold.
Let $q = \alpha + \beta + \gamma$, then the number of voters $n = 6 + 2q$. Since $\gamma = q - \alpha - \beta$, the two inequalities become

$$\beta > -2\alpha + q - 2, \quad \beta > -\frac{1}{2}\alpha + \frac{q}{2} - 2.$$ 

A profile is determined by $\alpha, \beta$. Any integer values of $\alpha, \beta$ with $0 \leq \alpha + \beta \leq q$ are allowed, i.e., supporting profiles correspond to lattice points in the triangle with vertices $(0, 0), (q, 0), (0, q)$ in the $\alpha, \beta$ plane.

The inequalities above divide the triangle into four regions corresponding to $A$ being both the IR and Plurality winner, neither the IR nor Plurality winner, and two regions where $A$ is the IR winner but not the Plurality winner.
The lattice points in $Q$ correspond to profiles where $A$ is neither the Plurality nor IR winner; the lattice points in $F$ correspond to profiles where $A$ is both; the lattice points in $T_1$ and $T_2$ correspond to profiles where $A$ is the IR winner, but not the Plurality winner.
Suppose there are 30 voters, then \( q = 12 \). There are 91 profiles, which correspond to the lattice points in the triangle with vertices 
\((0, 0), (12, 0), \) and \((0, 12)\).

Counting lattice points in the regions as above, we see that there are 21 profiles (23%) where \( A \) is the IR winner but not the Plurality winner. Further, there are 50 profiles (55%) where \( A \) is both the IR and Plurality winner, and 20 (22%) where \( A \) is neither the IR nor the Plurality winner.

Our original example \((11, 0, 5, 7, 0, 7)\) corresponds to \( \alpha = 7 \) and \( \beta = 0 \).
Note that if there are three candidates and $A$ is the CW, then if $A$ wins under Plurality, $A$ must also win under IR. The only way $A$ can lose in IR is if $A$ drops out first, but if $A$ wins under Plurality, this can’t happen.

Since IR does better than Plurality, is it a better method?

Instant Runoff does not satisfy **monotonicity**: It is possible for a winning candidate to become a losing candidate by moving up on some preference lists without moving down on any!

Plurality and Borda Count satisfy monotonicity.
Voting methods that use a scoring vector are called **positional voting methods**. For three candidates, it is easy to see that any such method is giving by a **positional scoring vector** $(1, s, 0)$, where $0 \leq s \leq 1$. Candidates get 1 point for a first-place ranking and $s$ points for a second-place ranking.

Borda count corresponds to $s = 1/2$. It’s easy to see the Plurality corresponds to $s = 0$. Antiplurality corresponds to $s = 1$. In a **multistage positional** voting method, we use a positional vector to find the scores of the candidates, drop the candidate(s) with the lowest score, then calculate the score again. Continue until all candidates left have the same score – the winner(s). For $s = 0$, this gives us Instant Runoff.
With three candidates, we call these “two-stage” positional methods.

In a new project with M. Castle, we are looking at monotonicity for two-stage positional elections using the “pairwise tallies” methods of Saari.

Saari conjectures that two-stage Borda count (TS) is least likely to have monotonicity violations while Instant Runoff (IR) and two-stage Antiplurality are most likely to have monotonicity violations.

We have found an example which shows that the relationship between the possibility of monotonicity for two-stage Borda and Instant Runoff is not as simple as we thought.
Some preliminary observations

It is easy to see that a monotonicity violation can occur only for cyclic profiles. We assume \( A \succ B \succ C \succ A \Rightarrow \) the essential profile is \((e_1, 0, e_3, 0, e_5, 0)\) with \(e_i + e_j > e_k\) for all \(i \neq j \neq k\).

The generic profile is then \((e_1 + \alpha, \beta, e_3 + \gamma, \alpha, e_5 + \beta, \gamma)\).

In our situation, if candidate \(X\) has the lowest score and drops first, then the winner is whichever of the remaining two candidates wins the pairwise comparison. This is because all scoring methods reduce to Majority Rule for two candidates. So if \(C\) drops first, then \(A\) wins since \(A \succ B\) ... and so on.

Note that for a reversal pair \((\alpha, \beta, \gamma, \alpha, \beta, \gamma)\), the Borda score of each candidate is \(\alpha + \beta + \gamma\) and hence the two-stage Borda winner is the same for every supporting profile. This is not true for other positional methods.
We may as well assume that $A$ is the two-stage Borda (TB) winner. This happens iff

$$2e_1 > e_3 + e_5, \quad e_1 + e_5 \geq 2e_3.$$  

These inequalities imply that $P(B, C) > P(C, A)$. We look at monotonicity in the following sense: Monotonicity is violated if $X$ is the winner and $X$ goes from winning to losing after some voters move $X$ from second place to first place on their preference list.

Consider TB monotonicity violations. In our case, this means $A$ moves from second to first place on some lists and loses. Hence the first candidate dropped changes from $C$ to $B$ and thus the only change in votes we need consider is $A$ gaining votes at the expense of $B$: $B\ A\ C$ becomes $A\ B\ C$. 

Vicki Powers  Emory University, Atlanta, GA  currently: National Science Foundation, Division of Mathematical Sciences  The Best Way to Choose a Winner
Assume the worst case for $A$, i.e., all $BAC$ voters switch to $ABC$. The new profile is $(e_1 + \alpha + \gamma, \beta, e_3 + \gamma, \alpha, e_5 + \beta, 0)$. Then $A$’s Borda score does not decrease, $C$’s score stays the same, and $B$’s score decreases by $\gamma/2$.

This means $A$ loses under the new profile iff $\gamma/2$ is large enough to make up the difference between the scores of $B$ and $C$ iff

$$\gamma > e_1 - 2e_3 + e_5.$$

Monotonicity will hold as long as $\gamma \leq e_1 - 2e_3 + e_5 > 0$. The worst case for $A$ is when $\gamma$ is as large as possible, so for a fixed $n$, the worst case is $\alpha = \beta = 0$.

Let $n$ be the number of voters. If $\gamma \leq e_1 - 2e_3 + e_5$, then $n = e_1 + e_3 + e_5 + 2\gamma$ implies

$$n \leq e_1 + e_3 + e_5 + 2(e_1 - 2e_3 + e_5) = 3(e_1 - e_3 + e_5) = 3P(B, C).$$
We have just proven: Under TB, no monotonicity violation can occur in any profile with the fixed essential profile iff $n \leq 3P(B, C)$.

Similarly, we can show that if $P(A, B) > P(C, A)$, under IR, no monotonicity violation occurs in any profile with the fixed essential profile iff $n \leq 2P(B, C) + P(C, A)$.

Since $P(B, C) > P(C, A)$, this means that the maximum number of voters with no monotonicity violation is higher for TB than for IR, in the case where $P(A, B) > P(C, A)$. 
Consider the essential profile \((4, 0, 3, 0, 2, 0)\), then \(P(A, B) = 5\), \(P(B, C) = 3\), and \(P(C, A) = 1\) and \(A\) is the TB winner.

The result above says that no monotonicity violation is possible for \(n \leq 9\) and for \(n = 11\) there is a monotonicity violation.

There are exactly three profiles with \(n = 11\), corresponding to \(\alpha = 1, \beta = 1,\) and \(\gamma = 1\). It is easy to check that for each, there is no monotonicity violation under IR. In this case, the threshold value for IR is higher than for TB.

Consider all essential profiles with exactly 9 voters for which our assumed conditions hold. There are three of these, with the following values for \((e_1, e_3, e_5)\): \((4, 1, 4)\), \((4, 2, 3)\), \((4, 3, 2)\). Our result says that the threshold values for MB for these three are 21, 15, and 9 respectively. The proposition says that the threshold values for IR for the first two are 17 and 11. It’s not too hard to see that the threshold value for IR for \((4, 3, 2)\) is 15.
Current and future work

Current projects include

- Looking at **proportional pairwise tallies** $P(X, Y)/(\# \text{ of voters})$ instead of $P(X, Y)$.

- Generalizing to 4 candidates. In this case, describing the set of supporting profiles from the essential profile is much more complicated.

- Use lattice-point counting to look at things like **Condorcet efficiency** for various voting methods: Given that there is a CW, what is the probability that the CW wins?
Thanks to the National Science Foundation and Universität Konstanz for making my trip and this talk possible.

Thanks for listening!