MINIMUM DEGREE CONDITIONS FOR A GRAPH TO BE PAN-$k$-LINKED

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Abstract. For a $k$-linked graph $G$ and a set $S$ of $2k$ distinct vertices of $G$, let $T$ denote the minimum order of a $k$-linkage for $S$ in $G$. A graph $G$ is said to be pan-$k$-linked if it is $k$-linked and for all sets $S$ of $2k$ distinct vertices of $G$, there exists a $k$-linkage of order $t$ for all $t$ such that $T \leq t \leq |V(G)|$. We show that for $k \geq 3$ and $n \geq 4k$, a graph on $n$ vertices satisfying $\delta(G) \geq \frac{n+2}{2}$ is pan-$k$-linked.

1. Introduction

Definition 1.1. A path-system $\mathcal{P}$ of $G$ is a family of vertex-disjoint paths $P_1, P_2, \ldots, P_k$ of $G$.

Definition 1.2. A graph $G$ is said to be $k$-linked if for every $2k$ distinct vertices $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$, $G$ has a path-system $\mathcal{P} = P_1, P_2, \ldots, P_k$ such that, for all $i$, $P_i$ is an $[a_i, b_i]$-path.

Definition 1.3. Let $S = \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ be a set of $2k$ vertices of $G$. We say that a path-system $\mathcal{P} = P_1, P_2, \ldots, P_k$ is a $k$-linkage for $S$ if for all $i$, $P_i$ is an $[a_i, b_i]$-path.

Definition 1.4. A graph $G$ is said to be pan-$k$-linked if it is $k$-linked and for all sets $S$ of $2k$ distinct vertices of $G$, there exists a $k$-linkage of order $t$ for all $t$ such that $T \leq t \leq |V(G)|$, where $T$ denotes the minimum order of a $k$-linkage for $S$ in $G$.

Definition 1.5. In a $k$-linked graph $G$, a $k$-linkage $\mathcal{P} = P_1, P_2, \ldots, P_k$ is called a proper $k$-linkage if

$$\sum_{i=1}^{k} |V(P_i)| < |V(G)|.$$  

Theorem 1.6. [1] If $G$ is a graph on $n \geq 4k$ vertices with $\sigma_2(G) \geq n + 2k - 3$, then $G$ is $k$-linked.

Theorem 1.7 (Williamson or Hendry?). If $G$ is a graph on $n$ vertices with $\delta(G) \geq \frac{n+2}{2}$, then $G$ is panconnected.

Theorem 1.8. [2] Let $S$ be an $H$-subdivision on $\overline{S}$, $R$ be a set of vertices not in $S$, and $A = V(S) \cup R$. Then, if $\delta(R, A) > \alpha(S) + |R| - 1$, then $R$ is insertible in $S$.

2. Results

Theorem 2.1. If $k \geq 3$, $n \geq 4k$, and $\delta(G) \geq \frac{n+2k-1}{2}$, then $G$ is $k$-linked in such a way that for every set $S$ of $2k$ distinct vertices of $G$, every proper $k$-linkage
\(\mathcal{P} = P_1, P_2, \ldots P_k\) for \(S\) can be extended to a \(k\)-linkage \(\mathcal{P}' = P_1', P_2', \ldots P_k'\) for \(S\) such that
\[
\sum_{i=1}^{k} |V(P_i')| = \left(\sum_{i=1}^{k} |V(P_i)|\right) + 1.
\]

Proof. Note that since \(\delta(G) \geq \frac{n+2k-1}{2} > \frac{n+2k-3}{2}\), \(G\) is \(k\)-linked by Theorem 1.6. If for every set \(S\) of \(2k\) distinct vertices of \(G\), we have that every proper \(k\)-linkage for \(S\) can be extended by one more vertex, then we are done. So, assume there exists a set \(S\) which has a proper \(k\)-linkage that cannot be extended by one vertex. Among all \(k\)-linkages for this \(S\) with this property, choose the \(k\)-linkage \(P = P_1, P_2, \ldots P_k\) such that \(\sum_{i=1}^{k} |V(P_i)|\) is minimized. We now wish to show that we can extend \(P\) by one vertex.

Let \(Q = G \setminus P\). Also, let \(p = \sum_{i=1}^{k} |V(P_i)|\) and \(q = |V(Q)|\). Note that no vertex in \(Q\) can be adjacent to two adjacent vertices on any \(P_i\) because otherwise we could extend our \(k\)-linkage by one more vertex.

Thus,
\[
\delta(Q, P) \leq \frac{p + k}{2}
\]
and for all \(w \in V(Q)\),
\[
deg_Q(w) \geq \frac{n + 2k - 1}{2} - \sum_{i=1}^{k} \alpha(P_i)
= \frac{n + 2k - 1}{2} - \sum_{i=1}^{k} \left\lfloor \frac{|V(P_i)|}{2} \right\rfloor
\geq \frac{n + 2k - 1}{2} - \sum_{i=1}^{k} \frac{|V(P_i)| + 1}{2}
= \frac{n + 2k - 1}{2} - \frac{p}{2} - \frac{k}{2}
\geq \frac{q + k - 1}{2}.
\]

Since \(k \geq 3\), we have that \(\frac{q + k - 1}{2} > \frac{q + k}{2}\). Consequently, by Theorem 1.7, \(Q\) is panconnected. Note that this degree condition also implies that for all \(x, y \in V(Q)\),
\[
d(x, y) \leq 2.
\]

Now if \(q < k\), then
\[
\delta(Q, P) \geq \frac{n + 2k - 1}{2} - (q - 1)
= \frac{n + 2k - 1 - 2q + 2}{2}
= \frac{p + q + 2k - 1 - 2q + 2}{2}
= \frac{p - q + 2k + 1}{2}
\geq \frac{p + k + 1}{2}.
\]

However, this contradicts equation 2.1. Thus, \(q \geq k + 1\).
We now wish to show that no vertex of $Q$ can be adjacent to two vertices $x, y$ on any $P_i$ of $P$ such that $2 \leq d_{P_i}(x, y) \leq q - 1$ (Note that the case where $d_{P_i}(x, y) = 1$ has been taken care of in the above argument). To show this, we will assume the opposite is true. That is, assume there exists a $w \in Q$ and a $P_i$ in $P$ such that $x, y \in N_{P_i}(w)$ and $2 \leq d_{P_i}(x, y) \leq q - 1$. Let $R$ be the path between $x$ and $y$ on $P_i$ (not including $x$ and $y$). We will assume that $x$ and $y$ are chosen so that $w$ has no neighbors on $R$. Let $r = |V(R)|$. Thus, we have $d_{P_i}(x, y) = r + 1$.

Assume first that $r \geq 4$. Let $u$ be a vertex of $R$. Either $d_{P_i}(u, x) \geq 3$ or $d_{P_i}(u, y) \geq 3$. Assume, without loss of generality, that the former is true. We know that $u$ is not adjacent to $w$. If $u$ has a neighbor $v \in Q \setminus \{w\}$, then by the panconnectedness of $Q$ we know there exists a $[v, w]$-path $R'$ in $Q$ of length $d_{P_i}(u, x) - 1$. Thus, we can extend our $k$-linkage by one vertex by removing the vertices of $R$ from $P_i$ and using the path $xwR'vy$ to complete $P_i$.

Therefore, we assume that no vertex of $R$ has a neighbor to $Q \setminus \{w\}$. So, for every vertex $u$ of $R$, $\deg_{Q}(u) = \deg_{P}(u) \geq \frac{n + 2k - 1}{2}$. Consider the path-system $B = B_1, B_2, \ldots, B_k, B_{k+1}$ where $B_j = P_j$ for $j \neq i, j \neq k + 1$ and $B_i, B_{k+1}$ are the two paths created from $P_i$ by removing the path $R$. We will now picture $B$ as an $H$-subdivision on $S$ (our set of $2k$ distinct vertices) where $H$ is the graph consisting of $k + 1$ independent edges. Let $A = V(B) \cup R = P$. Then since for all vertices $u$ of $R$, $\deg_{P}(u) \geq \frac{n + 2k - 1}{2}$, we have that

$$\delta(R, A) \geq \frac{n + 2k - 1}{2} = \frac{p + q + k + k - 1}{2} > \frac{p + r + k - 1}{2} = \frac{p - r + k + 1}{2} + r - 1 \geq \alpha(B) + |R| - 1.$$ 

Note here that we have used the facts that $q + k > r$ and $\alpha(B) = \lceil \frac{q - r + k}{2} \rceil$. So, by Theorem 1.8, $R$ is inseparable in $B$. Insert the vertices of $R$ into $B$. This leaves a $k + 1$ path-system $B'$ which has order $p$ and the same path endpoints as the path-system $B$. Thus, we may now use the edges $xw$ and $wy$ to form our $k$-linkage for $S$ of order $p + 1$.

Consequently, assume that $r \leq 3$.

\[ \square \]

**Corollary 2.2.** If $k \geq 3$, $n \geq 4k$, and $\delta(G) \geq \frac{n + 2k - 1}{2}$, then $G$ is pan-$k$-linked.

**References**
