Ore-type degree conditions, Path-Systems and Linkages in Graphs

Abstract

Let $G$ be a graph of order $n$ and $\sigma_2(G) = \min\{d(u) + d(v) : uv \notin E(G)\}$. It is easy to see that if $\sigma_2(G) \geq n - 1$, then $G$ is connected. Further, any pair $u, v \in V(G)$ of non-adjacent vertices are the end-vertices of a path of size two. Ore proved that if $\sigma_2(G) \geq n + 1$, then for any pair $u, v \in V(G)$ of distinct vertices, there is a $[u, v]$-path covering all vertices of $G$. In this paper, we generalize these results to higher connectivities, giving the minimum lower bounds on $\sigma_2(G)$ which ensure the existence minimal and maximal path-systems and linkages between sets of vertices.

1 Preliminaries

Unless otherwise specified, in this paper, $G = (V, E)$ will denote a simple loopless graph with $|V(G)| = n$. Let $K_n$ represent the complete graph (meaning, having all possible edges) on $n$ vertices. For $u \in V(G)$, let

$$N(u) = \{v \in V(G) : uv \in E(G)\},$$

$$d(u) = |N(u)|$$

and

$$\sigma_2(G) = \min\{d(u) + d(v) : uv \notin E(G)\}.$$  

A graph $G$ is said to be connected if for any two distinct vertices $u, v \in V(G)$, there is a $[u, v]$-path in $G$. It is easy to see that if a graph $G$ or order $n$ is such that $\sigma_2(G) \geq n - 1$ then $G$ must be connected. Indeed, take any two non-adjacent vertices $u$ and $v$ of $G$; since $d(u) + d(v) \geq n - 1 > |G - u - v|$, there must be a vertex $w \in N(u) \cap N(v)$, hence the connectivity of $G$.

The distance dist$(u, v)$ between two vertices $u$ and $v$ of a graph $G$ is defined to be the minimum size (number of edges) of a $[u, v]$-path. The diameter diam$(G)$ of $G$ is the maximum possible distance between two vertices of $G$. If $G$ is disconnected (not connected) we let diam$(G) = \infty$. The argument of the previous paragraph shows us that

**Fact 1** If $\sigma_2(G) \geq n - 1$, then diam$(G) \leq 2$. 

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Note that the condition $\sigma_2(G) \geq n - 2$ does not even ensure connectivity, as exemplified by a graph having two complete components. This shows that the lower bound $\sigma_2(G) \geq n - 1$ is best possible.

The graph $G$ is said to be Hamilton-connected if for any pair $(u, v)$ of vertices of $G$, there exists a Hamiltonian path between $u$ and $v$ (that is, a $[u, v]$-path covering all the vertices of $G$). Ore [2] proved that

**Theorem 1** If $\sigma(G) \geq n + 1$ then $G$ is Hamilton-connected.

The lower bound on $\sigma_2(G)$ is the best possible, as exemplified by a balanced complete bipartite graph (a graph constituted of two sets $X$ and $Y$ of $\frac{n}{2}$ vertices each, no edges inside $X$ or $Y$, but all edges between $X$ and $Y$).

A graph $G$ is said to be $k$-connected if one must remove at least $k$ vertices to either disconnect the graph, or leave only one vertex. In other words, $G$ is $k$-connected if for any set $S \subseteq V(G)$, if $G - S$ has only one vertex, or more than one component, then $|S| \geq k$. The connectivity $\kappa(G)$ of a graph $G$ is the maximum $k$ such that $G$ is $k$-connected. As a consequence of Menger’s famous theorem of 1927 [1], we have

**Theorem 2** A graph $G$ is $k$-connected if and only if for any pair $(A, B)$ of disjoint $k$-sets of $V(G)$, there are $k$ disjoint paths joining every vertex of $A$ to a vertex of $B$.

In light of the equivalence pointed out by Theorem 2, graph theorists were brought to the following alternate measure of connectivity:

**Definition 1** A graph $G$ is said to be $k$-linked if for every $2k$ distinct vertices $a_1, \ldots, a_k, b_1, \ldots, b_k$, $G$ contains $k$ disjoint paths $P_1, \ldots, P_k$ such that, for all $i$, $P_i$ is a $[a_i, b_i]$-path.

The objective of this paper is to generalize Fact 1 and Theorem 1 to $k$-connectivity (in the sense of the equivalence pointed out in Theorem 2) and $k$-linkages.

## 2 Generalization

Let $V_k(G)$ be the family of all $k$-tuples of vertices of a graph $G$.

Let $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ be a disjoint pair of elements of $V_k(G)$. An $(A, B)$-system $\mathcal{P}$ is a set of $k$ vertex-disjoint paths $P_1, \ldots, P_k$
joining the vertices of $A$ to the vertices of $B$ (i.e. for every $\lambda \in [k]$, there exists $i, j \in [k]$ such that $P_\lambda$ is a $[a_i, b_j]$-path.) An $(A, B)$-linkage is an $(A, B)$-system in which, for every $P_i \in \mathcal{P}$, $P_i$ joins $a_i$ to $b_i$. Thus an $(A, B)$-linkage is an $(A, B)$-system in which we get to specify the end vertices of the paths.

Let $\mathcal{S}(A, B)$ and $\mathcal{L}(A, B)$ denote the family of all $(A, B)$-systems and $(A, B)$-linkages of $G$ respectively. Let $V_k^2(G)$ be the family of all pairs $(A, B)$ of disjoint $k$-tuples of vertices of $G$.

The distance between two disjoint $k$-sets $A$ and $B$ of vertices may be defined in several natural ways, according to wether we take the size of the smallest $(A, B)$-system (resp. $(A, B)$-linkage), the size of the smallest possible path in an $(A, B)$-system (resp. $(A, B)$-linkage), or the size of the largest path in a minimum size $(A, B)$-system (resp. $(A, B)$-linkage). These different measures of distance imply different measures of diameters. Formally, for any disjoint $k$-sets $A$ and $B$ of vertices of a graph $G$, let

$$
\text{dist}_k(A, B) = \min_{P \in \mathcal{S}(A, B)} (|P| - k),
$$

$$
\overline{\text{dist}}_k(A, B) = \min_{P \in \mathcal{S}(A, B)} \max_{P \in \mathcal{P}} (|P| - 1),
$$

$$
\underline{\text{dist}}_k(A, B) = \min_{P \in \mathcal{S}(A, B)} \min_{P \in \mathcal{P}} (|P| - 1),
$$

$$
\text{diam}_k(A, B) = \max_{(A, B) \in V_k^2(G)} \text{dist}_k(A, B),
$$

$$
\overline{\text{diam}}_k(A, B) = \max_{(A, B) \in V_k^2(G)} \overline{\text{dist}}_k(A, B), \text{ and}
$$

$$
\underline{\text{diam}}_k(A, B) = \max_{(A, B) \in V_k^2(G)} \underline{\text{dist}}_k(A, B).
$$

The corresponding linked distances and diameters $\text{ldist}_k$, $\overline{\text{ldist}}_k$, $\underline{\text{ldist}}_k$, $\text{ldiam}_k$, $\overline{\text{ldiam}}_k$, $\underline{\text{ldiam}}_k$ are defined similarly, by replacing $\mathcal{S}(A, B)$ with $\mathcal{L}(A, B)$.

Note that saying that $G$ is $k$-connected is equivalent to saying that $\text{diam}_k(G) < \infty$ (or equivalently, $\overline{\text{diam}}_k(G) < \infty$). Similarly, saying that $G$ is $k$-linked is equivalent to saying that $\text{ldiam}_k(G) < \infty$ (or equivalently, $\overline{\text{ldiam}}_k(G) < \infty$).

We say that $G$ is Hamilton $k$-connected (resp. $k$-linked) if for any $(A, B) \in V_k^2(G)$, there is a $\mathcal{P}$ in $\mathcal{S}(A, B)$ (respectively, in $\mathcal{L}(A, B)$) such that $\mathcal{P}$ covers all the vertices of $G$. 

3
3 Results

One may easily see that $\sigma_2(G) \geq n + k - 2$ implies $\kappa(G) \geq k$. Indeed, this is the contrapositive of

$$\kappa(G) \leq k - 1 \implies \sigma_2(G) \leq n + k - 3 \quad (1)$$

which can be seen to be true since if $C$ is a cut set of order $k - 1$ and $A$ and $B$ were two components of $G - C$, then taking two vertices $x \in A$ and $y \in B$, we see that $xy \in E(G)$ yet

$$d(x) + d(y) \leq (|A| - 1) + |C| + (|B| - 1) + |C| \leq n + k - 3.$$

By considering the case where $A$ and $B$ are the only components of $G$, and both $(A \cup C)$ and $(B \cup C)$ induce complete graphs, we see that $\sigma_2(G) = n + k - 3$, yet $\kappa(G) = k - 1$, so the bound on $\sigma_2(G)$ is the best possible.

We show that in fact $\sigma_2(G) \geq n + k - 2$ implies $\text{diam}_k(G) \leq 2k$. This diameter is essentially the lowest possible in the sense that, in order to reduce it further, one must have a graph that is nearly complete, but the actual lowest possible $k$-diameter of a graph is $k$, so for completeness, we include the bounds on $\sigma_2(G)$ implying lower diameters than $2k$.

**Theorem 3** Let $G$ be a graph of order $n \geq 2k$ and $l \in [k]$. The following table relates the value of $\sigma_2(G)$ to the lowest upper bound on the $k$-diameter of $G$.

<table>
<thead>
<tr>
<th>$\sigma_2(G)$</th>
<th>$\text{diam}_k(G)$</th>
<th>$\text{diam}_k(G)$</th>
<th>$\text{diam}_k(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq n + k - 3$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$n + k - 2 \leq \sigma_2(G) \leq 2n - 2k - 2$</td>
<td>$2k$</td>
<td>$2$</td>
<td>$2$</td>
</tr>
<tr>
<td>$\sigma_2(G) = 2n - 2k - 2 + l$</td>
<td>$2k - l$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$2n - k - 2 \leq \sigma_2(G)$</td>
<td>$k$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

We see that the minimum bound on $\sigma_2(G)$ ensuring $\text{diam}_k(G) < \infty$ ($k$-connectivity) is $n + k - 2$, and when this happens, we have automatically the small diameter of $2k$. Then, until $2n - 2k - 1$ we cannot lower the diameter further. At $2n - k - 2$ we attain the smallest possible diameter $\text{diam}_k(G) = k$ (equivalently, $\text{diam}_k = \text{diam}_k = 1$).

Note that $\sigma_2(G)$ can larger than $2n - 4$, and that those graphs $G$ for which $\sigma_2(G) = 2n - 4$ have the property that a vertex cannot have more than one
non-adjacency. Hence these graphs are isomorphic to $K_n - M_m$ where $M_m$ is a set of $m$ independent edges of $K_n$ for some $m \in \lfloor n/2 \rfloor$. The following Theorem relates the value of $\sigma_2(G)$ to the linked-diameters of $G$. Since even $\sigma_2(G) = 2n - 4$ is not sufficient to force $ldiam_k(G) < 2k$, we include the linked-diameters of the $K_n - M_m$ graphs.

**Theorem 4** Let $G$ be a graph of order $n \geq 4k$ and $l \in [k]$. Let $M_{k-l}$ be a set $k-l$ independent edges of a complete graph $K_n$. The following table relates the value of $\sigma_2(G)$ to the lowest upper bound on the linked-diameters of $G$.

<table>
<thead>
<tr>
<th>$\sigma_2(G)$ $\leq n + 2k - 4$</th>
<th>$ldiam_k(G) \leq$</th>
<th>$ldiam_k(G) \leq$</th>
<th>$ldiam_k(G) \leq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_2(G) = n + 2k - 4 + l$</td>
<td>$3k - l$</td>
<td>$3$</td>
<td>$2$</td>
</tr>
<tr>
<td>$n + 3k - 4 \leq \sigma_2(G)$</td>
<td>$2k$</td>
<td>$2$</td>
<td>$2$</td>
</tr>
<tr>
<td>$G = K_n - M_{k-1}$</td>
<td>$2k - l$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$G = K_n$</td>
<td>$k$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

In order to generalize Theorem 1 we prove that

**Lemma 1** If $n \geq 4k$, and $\sigma_2(G) \geq n + k$, then, for any disjoint $k$-tuples $A, B \in V_k(G)$, if there exists an $(A, B)$-linkage, then there exists a Hamiltonian $(A, B)$-linkage. On the other hand, if $\sigma_2(G) < n + k$, $G$ may have $(A, B)$-systems which may not even be extended to Hamiltonian $(A, B)$-systems.

Using this Lemma, we see that Theorems 3 and 4 have the following corollaries:

**Corollary 5** If $n \geq 2k$ and $\sigma_2(G) \geq n + k$ then $G$ is Hamiltonian $k$-connected.

**Corollary 6** If $k \geq 2$, $n \geq 4k$, and $\sigma_2(G) \geq n + 2k - 3$, then $G$ is Hamiltonian $k$-linked.

### 4 Proof of Theorems

For two given subgraphs $F$ and $H$ of $G$, we denote by $F_H$ the subgraph of $G$ induced by the vertices of $F$ in $H$ (i.e. the graph with vertex set $V(F)$ and edge set $E(F) \cap E(H)$).
For any given subgraphs $A$ and $B$ of $G$, let $N(A)$ be the set of vertices of $G$ that are adjacent to at least one vertex of $A$, let $E(A, B)$ be the set of edges that have an end vertex in $A$ and the other in $B$, let $d(A, B) = |E(A, B)|$ (so for a given vertex $v \in V(A)$, $d(v, B) = N(v) \cap V(B)$, let $\delta(A) = \min_{v \in V(A)}\{d(v)\}$ let $\delta(A, B) = \min_{v \in V(A)}\{d(v, B)\}$.

Let $P = z_1 \cdots z_m$ be a path of $G$ and $z = z_i$ be a vertex of $V(P)$. If $i \geq 2$, the predecessor $z^-$ of $z$ is the vertex $z_{i-1}$. If $i \leq n - 1$, the successor $z^+$ of $z$ is the vertex $z_{i+1}$. For two vertices $z_i, z_j \in V(P)$, we let $[z_i, z_j]_P$ (or simply $[z_i, z_j]$ when the context is clear) be the subpath of $P$ between $z_i$ and $z_j$.

Let $K_n$ be the complete graph on $n$ vertices. If $A$ and $B$ are two disjoint graphs, let $A + B$ represent the graph having vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B)$ along with all possible edges between $A$ and $B$.

**Fact 2** Let $G$ be a graph, $P$ be a $[u, v]$-path of $G$, and $w \in V(G - P)$. If $d(w, P) \geq \left\lceil \frac{|P| + 1}{2} \right\rceil$, then $P$ may be extended to a $[u, v]$-path $P'$ of order $|P| + 1$.

Proof: Let $P = \{x_1, \ldots, x_p\}$. If $d(w, P) \geq \left\lceil \frac{p + 1}{2} \right\rceil$, then there is an $i \in [p]$ such that $wx_i, wx_{i+1} \in E(G)$, thus the $[x_1, x_p]$-path

$$P' = [x_1, x_i]x_iwx_{i+1}[x_{i+1}, x_p]$$

which we predicted.$\square$

**Proof of Theorem 3:**

Suppose $G$ satisfies the conditions of Theorem 3 and take any $(A, B) \in V_k(G)$ where $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$. Let $e_1, \ldots, e_s$ be a maximal independent set of edges of $E(A, B)$; without loss of generality, we may assume that $e_1 = a_1b_1, \ldots, e_s = a_sb_s$. Let $A' = \{a_1, \ldots, a_s\}$ and $B' = \{b_1, \ldots, b_s\}$.

If $s = k$, we are done, so assume $s \leq k - 1$. Now

$$d(a_t, A) \leq |A| - 1 = k - 1, \quad d(b_t, B) \leq |B| - 1 = k - 1,$$

(2)

and

$$d(a_t, B) + d(b_t, A) \leq s,$$

(3)

since $E(A - A', B - B') = \emptyset$ and if $d(a_t, B') + d(b_t, A') > s$, there were a $1 \leq t' \leq s$ such that $a_tb_{t'}, b_ta_{t'} \in E(G)$, thus replacing $a_tb_{t'}$ with these two
edges, we would contradict the maximality of the independent set of edges we started with. Also, for all \( s+1 \leq t \leq k \), \( a_t b_t \notin E(G) \), so \( d(a_t) + d(b_t) \geq \sigma_2(G) \).

Hence, if \( \sigma_2(G) \geq n + k - 2 \) then

\[
d(a_t, G - A - B) + d(b_t, G - A - B) \geq (n + k - 2) - (d(a_t, A) + d(a_t, B) + d(b_t, A) + d(b_t, B)).
\]

So by (2) and (3),

\[
d(a_t, G - A - B) + d(b_t, G - A - B) \geq (n + k - 2) - 2(k - 1) - s = (n - 2k) + (k - s),
\]

which, since \( |G - A - B| = n - 2k \), shows that \( |N(a_t, G - A - B) \cap N(b_t, G - A - B)| \geq k - s \). This ensures that there are \( (k - s) \) distinct vertices \( z_{t+1}, \ldots, z_k \) of \( G - A - B \) such that \( z_t \) is adjacent to both \( a_t \) and \( b_t \) for \( s + 1 \leq t \leq k \), thus the existence of the required \((A, B)\)-system.

The \((A, B)\)-system constructed verifies \( \text{diam}_k(G) \leq 2 \), thus \( \text{dist}_k(A, B) \leq 2k \). Since the pair \((A, B)\) was arbitrary, \( \text{diam}(G) \leq 2 \) and \( \text{diam}_k(G) \leq 2k \).

If \( \sigma_2(G) \geq 2n - 2k - 2 + l \), where \( l \in [k] \), then for all \( s + 1 \leq t \leq k \),

\[
d(a_t, B) + d(b_t, A) \geq (2n - 2k - 2 + l) - (d(a_t, G - A - B) + d(b_t, G - A - B) + d(a_t, A) + d(b_t, B)),
\]

thus, since

\[
d(a_t, G - A - B) + d(b_t, G - A - B) \leq 2|G - A - B| = 2n - 4k,
\]

using (2), we get

\[
d(a_t, B) + d(b_t, A) \geq (2n - 2k - 2 + l) - (2n - 4k + 2(k - 1)) = l
\]

By (3) then, we get \( s \geq l \), which shows that \( \text{diam}(G) = 1 \) and \( \text{diam}(G) \leq 2k - l \).

We have already seen, in section 3 that \( \sigma_2(G) \leq n + k - 2 \) is the lowest bound implying \( k \)-connectivity, or equivalently a finite diameter. To see that \( \sigma_2(G) = 2n - 2k - 2 + l \) is the smallest value of \( \sigma_2(G) \) implying \( \text{diam}_k(G) \leq 2k - l \), consider the complete graph \( K_n \), and two disjoint \( k \)-tuples \( A \) and \( B \).
and a subset $B_{k-l+1} \subset B$ of order $k-l+1$. Then the graph $G = K_n - E_{K_n}(A, B_{k-l+1})$ verifies $\sigma_2(G) = 2n - 2k - 3 + l$, yet $\text{dist}_k(A, B) = 2k - l + 1$. By letting $l = 1$ we also see that $\sigma_2(G) = 2n - 2k - 2$ is not enough to yield $\text{diam}_k(G) = 1$ of $\text{diam}_k(G) < 2k$.

\[ \square \]

**Proof of Theorem 4:**

Theorem 4 is a consequence of the three following Claims.

**Claim 1** If $G$ is a graph on $n \geq 4k$ vertices and $\sigma_2(G) \geq n + 2k - 3$ then $\text{ldiam}_k(G) \leq 3$.

Let $G$ be a graph such that $\sigma_2(G) \geq n + 2k - 3$ (6) where $k$ is an integer such that $n \geq 4k$. Let $S = \{a_1, \cdots, a_k, b_1, \cdots, b_k\}$ be any set of $2k$ disjoint, $A = (a_1, \cdots, a_k)$, and $B = (b_1, \cdots, b_k)$. Let $\mathcal{P} = \{P_1, \cdots, P_{k'}\}$ be a family of paths linking $k'$ vertices of $A$ to the corresponding vertices of $B$, where all paths have order no more than 4, and without loss of generality we assume that for every $i$ (1 $\leq i \leq k'$), $P_i$ links $a_i$ to $b_i$, and that for some non-negative integers $t_1, t_2, t_3, t_4$ (with $t_1 + t_2 + t_3 = k'$), $|P_1| = \cdots = |P_{t_1}| = 2$, $|P_{t_1+1}| = \cdots = |P_{t_1+t_2}| = 3$, and $|P_{t_1+t_2+1}| = \cdots = |P_{t_1+t_2+t_3}| = 4$. We choose $\mathcal{P}$ so that $k' = |\mathcal{P}|$ is maximal, (7) and under this condition, $\Sigma^k_{i=1} |P_i|$ is minimal, (8) and under this condition, $\max_{i > k'} \min_{i \geq k'} \{d(a_i, G - S - V(\mathcal{P})), d(b_i, G - S - V(\mathcal{P}))\}$ is maximal. (9)

Let $R = V(G) - S$. By (8), for every $i$ ($t_1 + 1 \leq i \leq k$), $a_i b_i \notin E(G)$ so that

$\begin{align*}
d(a_i, S), d(b_i, S) &\leq |S| - 2 = 2k - 2, \quad \text{thus} \\
d(a_i, S) + d(b_i, S) &\leq 4k - 4.
\end{align*}$

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Note that if \( k' = k \) then we have our result, so assume \( k' < k \) (i.e. \( t_4 \geq 1 \)) and let \( S_1 = S \cap \bigcup_{i=1}^{t_1} V(P_i) \), \( S_2 = S \cap \bigcup_{i=t_1+1}^{t_2} V(P_i) \), \( S_3 = S \cap \bigcup_{i=t_1+t_2+1}^{t_3} V(P_i) \), and \( S_4 = S - S_1 - S_2 - S_3 \). Let \( R_2 = R \cap \bigcup_{i=t_1+t_2+1}^{t_3} V(P_i) \), \( R_3 = R \cap \bigcup_{i=t_1+t_2+t_3+1}^{t_4} V(P_i) \), and \( R_4 = R - R_2 - R_3 \).

Let \( u = a_j \) and \( v = b_j \) where \( k'+1 \leq j \leq k \) is such that \( \min \{ d(u, R_4), d(v, R_4) \} = \max_{i > k'} \min \{ d(a_i, R_4), d(b_i, R_4) \} \). Let \( \alpha = d(u, R_4) \) and \( \beta = d(v, R_4) \) and assume, without loss of generality, that \( \alpha \leq \beta \).

Note that

\[
d(\{u, v\}, R_3) \leq 2t_3
\]

(12)
since otherwise there would be an \( t_1 + t_2 + 1 \leq i \leq t_1 + t_2 + t_3 \) such that \( d(\{u, v\}, P_i \cap R) \geq 3 \), implying that one of the two vertices \( w \) of \( P_{i-a_i-b_i} \) is adjacent to both \( u \) and \( v \). Yet then the path \( P_i \) of order 4 may be replaced with the path \( uwu \) of order 3, contradicting the minimality (8).

**Case 1:** Assume \( \alpha \geq 1 \). Then let \( x \) and \( y \) be any vertices of \( N(u, R_4) \) and \( N(v, R_4) \) respectively.

We prove a few upper bounds on the number of edges between vertices \( u, v, x \) and \( y \), and different parts of the graph. First of all,

\[
d(\{x, y\}, S_2 \cup R_2) + d(\{u, v\}, R_2) \leq 6t_2.
\]

(13)

Indeed, if this isn’t the case, then for some \( t_1 + 1 \leq i \leq t_1 + t_2 \), we must have \( d(\{x, y\}, P_i) + d(\{u, v\}, P_i \cap R) \geq 7 \). Note that

\[
|\{x, y\}| \cdot |P_i| + |\{u, v\}| \cdot |P_i \cap R| = 8,
\]

so there is at most one missing edge. Let \( P_i = a_iwb_i \). If edge \( uw \) is missing then

\[
\mathcal{P}' = (\mathcal{P} - P_i) \cup vwxu \cup a_iyb_i
\]

contradicts the maximality (7). One may verify that every other case of a missing edge leads to a similar situation where one may find two disjoint paths; a \( (u, v) \)-path of order 3 and an \( (a_i, b_i) \)-path of order 3, contradicting (7).

Further,

\[
d(\{x, y\}, S_3 \cup S_4) \leq 2(t_3 + t_4)
\]

(14)
as if this were not true, there would be an \( t_1 + t_2 + 1 \leq i \leq k \) such that \( d(\{x, y\}, \{a_i, b_i\}) \geq 3 \), ensuring the existence of the path \( a_ixb_i \) (or \( a_iyb_i \)) of order 3. If \( t_1 + t_2 + 1 \leq i \leq t_1 + t_2 + t_3 \), this contradicts (8), and if \( t_1 + t_2 + t_3 + 1 \leq i \leq k \), this contradicts (7).
Since \(|S_1| = 2t_1\) and \(|R_3| = 2t_3\) we have

\[
d(\{x, y\}, S_1 \cup R_3) \leq 4(t_1 + t_3).
\]

(15)

Finally, if \(d(x, N(v) \cap R_4) \neq 0\) or \(d(y, N(u) \cap R_4) \neq 0\), then (7) would be contradicted, so

\[
d(x, R_4) \leq |G - x| - |S| - |R_2| - |R_3| - |N(v, R_4)|
= n - 1 - 2k - t_2 - 2t_3 - \beta
\]

(16)

and

\[
d(y, R_4) \leq |G - y| - |S| - |R_2| - |R_3| - |N(u, R_4)|
= n - 1 - 2k - t_2 - 2t_3 - \alpha.
\]

(17)

One may verify that

\[
d(x) + d(y) + d(u) + d(v) \leq
\]

\[
d(\{u, v\}, S)
+ d(\{u, v\}, R_3)
+ d(\{x, y\}, S_2 \cup R_2)
+ d(\{x, y\}, S_3 \cup S_4)
+ d(x, R_4)
\]

Using (11), (12), (13), (14), (15), (16) and (17), we see that

\[
d(x) + d(y) + d(u) + d(v) \leq
\]

\[
4k - 4
+ 2t_3
+ 6t_2
+ 2(t_3 + t_4)
+ 4(t_1 + t_3)
+ n - 1 - 2k - t_2 - 2t_3 - \beta
\]

Simplifying this expression, and using the fact that \(t_1 + t_2 + t_3 + t_4 = k\), we see that

\[
d(x) + d(y) + d(u) + d(v) \leq 2n - 6 + 4k - 2t_4.
\]

Since \(uy, vx \notin E(G)\), our degree sum condition (6) shows on the other hand that

\[
d(x) + d(y) + d(u) + d(v) \geq 2n + 4k - 6.
\]
This shows that we must have $t_4 = 0$, a contradiction.

**Case 2:** Assume $\alpha = 0$. First we show that $\beta \geq 3$. Indeed, $uv \notin E(G)$, so using (12) we get

$$d(u, R_4) + d(v, R_4) \geq n + 2k - 3 - d(\{u, v\}, S)$$
$$- |N(\{u, v\}, R_2)| - d(\{u, v\}, R_3)$$
$$\geq n + 2k - 3 - (4k - 4) - 2t_2 - 2t_3$$
$$= n - 2k + 1 - 2(t_2 + t_3),$$

and since $t_4 \geq 1$, $t_2 + t_3 \leq k - 1$, hence using the fact that $n \geq 4k$ and $d(u, R_4) = 0$, we have

$$\beta = d(v, R_4) \geq n - 4k + 3 \geq 3.$$

Let $y$ be a vertex of $N(v, R_4)$. Note that

$$d(u, R_2) + d(y, S_2) \leq 2t_2$$

(18)
since otherwise, for some $t_1 + 1 \leq i \leq t_1 + t_2$ we would have $d(u, P_i \cap R) + d(y, P_i \in S) \geq 3$, implying that $ya_i, yb_i, uw \in E(G)$ where $w$ is the middle vertex of $P_i$. But then replacing $P_i$ with the path $a_i yb_i$, we obtain a system of paths satisfying conditions (7) and (8), but contradicting (9) since $u$ is adjacent to $w$ and $v$ is still adjacent to at least 2 vertices of $G - S - V(\mathcal{P})$. Further,

$$d(u, R_3) + d(y, R_3 \cup S_3) \geq 4t_3.$$

(19)

Indeed, if this were not the case, for some $t_1 + t_2 + 1 \leq i \leq t_1 + t_2 + t_3$, we would have $d(u, P_i \cap R) + d(y, P_i \geq 5$. Since we cannot have both $ya_i \in E(G)$ and $yb_i \in E(G)$ (or (8) would be contradicted), this shows that letting $P_i = a_i w zb_i$, we have $yw, yz, uw, uz \in E(G)$, and without loss of generality, $yb_i \in E(G)$. Replacing $P_i$ by the path $a_i wyb_i$ one may verify that we again contradict (9). Finally,

$$d(y, S_4) \leq t_4$$

(20)
or there would be a $t_1 + t_2 + t_3 + 1 \leq i \leq k$ with $xa_i, xb_i \in E(G)$, hence a path $a_i xb_i$ contradicting (7).
Now
\[ d(u) + d(y) = d(u, S) + d(u, R_2) + d(y, S_2) + d(u, R_4) + d(u, R_3) + d(y, S_3 \cup R_3) + d(u, R_2 \cup R_4) + d(y, S_1) + d(y, S_4) \]

Using (10), (18), (?, ?), (17), we find that
\[ d(u) + d(v) \leq (2k - 2) + 0 + 2t_2 + 4t_3 + (n - 1 - 2k - 2t_3) + 2t_1 + t_4 \]
\[ = n - 3 + 2(t_1 + t_2 + t_3 + t_4) - t_4 \]
\[ = n + 2k - 3 - t_4 \]
\[ < n + 2k - 3, \]
since \( t_1 + t_2 + t_3 + t_4 = k \) and \( t_4 \geq 1 \). Yet since \( uy \notin E(G) \) this contradicts (6).

Hence \( t_4 = 0 \), so \( G \) is \( k \)-linked and since we required all paths of \( P \) to be of order smaller or equal to 4, we see that, in fact, \( \text{ldiam}(G) \leq 3 \).

**Claim 2** Let \( G \) be a graph of order \( n \), \( k \) be a positive integer such that \( n \geq 4k \) and \( l \) be a positive integer with \( 1 \leq l \leq k \). If \( \sigma_2(G) \geq n + 2k + l - 4 \) then \( \text{ldiam}_k(G) \leq 3k - l \).

**Proof:** Let \( G \) be a graph satisfying the conditions of the Claim. Let \( S, R, A, B, P, t_1, t_2, t_3 \) and \( t_4 \) be defined as in the proof of Claim 1. The said Claim shows that \( t_4 = 0 \), so that \( k = t_1 + t_2 + t_3 \). If \( t_1 + t_2 \geq l \), then
\[ |P| = 2t_1 + 3t_2 + 4t_3 \]
\[ = 4(t_1 + t_2 + t_3) - (t_1 + t_2) - t_2 \]
\[ \leq 4k - l, \]
which implies \( \text{ldiam}_k(G) = |P| - k \leq 3k - l \), which is to be proven. Hence we assume
\[ t_2 + t_3 \leq l - 1. \] (21)

Now for every \( t_1 + t_2 + 1 \leq i \leq k \) we have \( a_i b_i \notin E(G) \), so
\[ d(a_i, b_i; P - S) \geq \sigma_2(G) - 2(2k - 2) \]
\[ \geq n - 2k + l \]
\[ = |G - S| + l, \]
implying that there are at least \( l \) vertices in \( G \) which are adjacent to both \( a_i \) and \( b_i \). The minimality of \(|\mathcal{P}|\) implies that none of these vertices may be in \( P_i \) or in \( G - \mathcal{P} \) since otherwise a \((a_i, b_i)\)-path of order four could be replaced by a path of order three. Also, by (21), at least one of these vertices must be in \( P_j - \{a_j, b_j\} \), where \( t_1 + t_2 + 1 \leq j \leq k \) and \( j \neq i \).

Let \( D \) be a digraph of order \( t_3 \) obtained by taking \( P_{t_1+t_2+1}, \ldots, P_k \) to correspond to the vertices, and where there is an edge from \( P_i \) to \( P_j \) \((i \neq j)\) if and only if there is a vertex \( w \) in \( P_j - \{a_j, b_j\} \) such that \( a_iw, bw \in E(G) \).

One may easily verify that if \( D \) had a directed cycle then one could replace every path \( P_i \) of order 4 corresponding to the vertices of this directed cycle with an \((a_i, b_i)\)-path of order 3, hence contradicting the minimality of \(|\mathcal{P}|\). Yet the previous paragraph implies that every vertex of \( D \) has at least one edge coming out of it, and this can be seen to imply the existence of a directed cycle in \( D \) (note that we allow this cycle to be of order two).

Indeed, take the last vertex \( z \) of a directed path \( Z \) of \( D \) of maximal order. Since \( z \) must be adjacent to a vertex \( z' \) of \( D \), and that \( z' \) cannot be in \( D - Z \), or the maximality of \( Z \) would be contradicted, we see that \( z' \) must be in \( Z \), creating a directed cycle in \( D \), and hence completing the proof of our Theorem. \( \square \)

**Claim 3** The lower bounds on \( \sigma_2(G) \) in Claim 1 and Claim 2 are minimal.

**Proof:** To see that the lower bound on \( \sigma_2(G) \) in Claim 4 is the smallest possible yielding \( \text{ldiam}_k(G) < \infty \), we construct a graph \( G(k) \) verifying \( \sigma_2(G) = n + 2k - 4 \) and which is not \( k \)-linked. Indeed, take the complete graph \( K_{n-k} \), \( B \) and \( C \) be any two disjoint subgraphs of \( K_{n-k} \) of order \( k \) and \( k-1 \) respectively (note that \( n \) is assumed to be greater or equal to \( 4k \)). The vertices of \( B \) will be labeled \( b_1, \ldots, b_k \). Let \( A \) be a complete graph on \( k \) vertices having vertices labeled \( a_1, \ldots, a_k \). Consider the graph

\[
G(k) = (A + K_{n-k}) - E(A, K_{n-k} - (B \cup C)) - \{a_1b_1, \ldots, a_kb_k\}.
\]

of order \( n \). The only non-adjacencies are between \( A \) and \( K_{n-k} \), yet if we take any vertex \( a \in V(A) \) and \( z \in V(K_{n-k}) \) we find that

\[
d(a) + d(z) \geq d(a, A) + d(a, B) + d(a, C) + d(z, K_{n-k}) \geq (k - 1) + (k - 1) + (k - 1) + n - k - 1 = n + 2k - 4,
\]

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and if we take \( z \) to be in \( V(K_{n-k} - B - C) \), we have equality. This shows that \( \sigma_2(G) = n + 2k - 4 \). Yet since the only edges of \( E(A, B) \) that could be used for our linking paths are the missing \( a_ib_i \) edges, any path linking the vertices of \( A \) to those of \( B \) must use at least one vertex of \( C \). Since \( |C| = k - 1 \), there is no \((A, B)\)-linkage in \( G(k) \).

The condition on \( \sigma_2(G) \) of Claim 2 can be seen to be the best possible. Consider for example the graph

\[
G(k, l) = (A + K_{n-k} - E(B, C) - E(A, K_{n-3k} - B - C - L) - \{a_1b_1, \ldots, a_kb_k\},
\]

where \( A \) is a complete graph of order \( k \), disjoint from \( K_{n-k} \), and whose vertices are labeled \( a_1, \ldots, a_k \), \( B \) is a subgraph of \( K_n - k \) with vertices labeled \( b_1, \ldots, b_k \), and \( C \) and \( L \) are two disjoint subgraphs of \( K_{n-k} - B \) or order \( k \) and \( l \) respectively.

Since we removed the edges \( a_1b_1, \ldots, a_kb_k \), the edges of \( E_{G(k,l)}(A, B) \) cannot be used in a \((A, B)\)-linkage \( P \), and all paths of \( P \) must be of order at least 3. The only edges left from \( A \) to the rest of the graph \( G(k, l) \) are those of \( E(A, C \cup L) \). Yet since we removed all the edges between \( B \) and \( C \), the only paths of \( |P| \) that have order 3 must go through \( L \). Since \( |L| = l \), there can be no more than \( l \) such paths, thus \( \Sigma_{P \in \mathcal{P}} |P| \geq 4(k - l) + 3l = 4k - l \), implying that \( \text{ldiam}_k(G(k, l)) \geq 3k - l \).

Yet let \( u \) and \( v \) be any two non-adjacent vertices of \( G(k, l) \). If \( u \in B \) and \( v \in C \), then

\[
d(u) + d(v) = d(u, G - A - C) + d(u, A) + d(v, G - A - C) + d(v, A) \\
= (n - k - k - 1) + (k - 1) + (n - k - k - 1) + k \\
= 2n - 2k - 3 \\
\geq n + 2k + l - 4
\]
since \( n \geq 4k + l + 1 \). If \( u \in A \) and \( v \in B \) then

\[
d(u) + d(v) = d(u, A) + d(u, B) + d(u, C \cup L) + d(v, G - A - C) + d(v, A) \\
= (k - 1) + (k - 1) + (k + l) + (n - k - k - 1) + (k - 1) \\
= n + 2k + l - 4.
\]

These are without loss of generality, the only possibilities for \( u \) and \( v \), thus \( \sigma_2(G(k, l)) = n + 2k + l - 4 \).

The tightness of Claim 2 is then illustrated by the graph \( G(k, l-1) \) which verifies \( \sigma_2(G(k, l-1)) = n + 2k + l - 5 \) yet \( \text{ldiam}_k(G(k, l)) \geq 3k - l \). \( \square \)
This concludes the proof of Theorem 4 □.

Before we prove the extension theorem, we will prove the following useful lemma:

Theorems 5 and 6 are a direct consequence of Theorems 3 and 4 using the following Lemma:

The Extension Lemma:

**Proof of The Extension Lemma:** Let \( G \) satisfy the conditions of the Lemma and \( (A, B) \in V_k(G) \) be such that there is an \((A, B)\)-linkage in \( G \). Let \( A = \{a_1, \ldots, a_k\} \), \( B = \{b_1, \ldots, b_k\} \) and \( \mathcal{P} = \{P_1, \ldots, P_k\} \) be an \((A, B)\)-linkage of maximal order. Let \( q = |Q| \), \( p = |P| \), and for every \( 1 \leq i \leq k \), let \( p_i = |P_i| \). Let \( P = (\mathcal{P})_G \) and \( Q = G - P \).

Note that for any \( u \in Q \) and \( P_i \in \mathcal{P} \), if \( z \in N(u, P_i) \), then \( z \notin N(u, P_i) \) or replacing \( P_i \) with \([a_i, z]_{P_i} \cup zu \cup uz^+ \cup [z^+, b_i]_{P_i} \) we would contradict the maximality of \( \mathcal{P} \). This implies that \( d(u, P_i) \leq \left\lfloor \frac{p + k}{2} \right\rfloor \), thus

\[
d(u, P) \leq \left\lfloor \frac{p + k}{2} \right\rfloor,
\]

which in turn yields that for any two vertices \( u \) and \( v \) of \( Q \),

\[
d(w, Q) + d(w', Q) \geq n + k - (p + k) = q,
\]

showing that \( Q \) is connected, so there is a \((u, v)\)-path in \( Q \), implying that in fact

\[
d(u, P) + d(v, P) \leq \left\lfloor \frac{p + k}{2} \right\rfloor, \tag{23}
\]

or again, the maximality of \( \mathcal{P} \) would be contradicted.

The fact that \( \sigma_2(G) \geq n + (k + 2) - 2 \) shows by Theorem 3 that \( G \) is \((k + 2)\)-connected. Thus \( |N(Q, P)| \geq k + 2 \), so by the pigeon-hole principal, some member of \( \mathcal{P} \), without loss of generality \( P_1 \), verifies \( |N(Q, P_1)| \geq 2 \).

Let \( x \) and \( y \) be such that \( \{x, y\} \in N(Q, P_1) \), \( y \) appears after \( x \) in the \( P_1 \), and \([x, y]_{P_1}\) is minimal. Let \( u, v \in V(Q) \) be such that \( ux, vy \in E(G) \) and \( R = [x^+, y^-]_{P_1} \). We cannot have \( y = x^+ \) or the maximality of \( \mathcal{P} \) would be contradicted, so \( R \neq \emptyset \). Let \( r = |R| \), \( P'_1 = [a_1, x]_{P_1} \) and \( P''_1 = [y, b_1]_{P_1} \).

By the minimality of \( [x, y]_{P_1} \), \( d(S, R) = 0 \), so the inequality (23), when applied to \( \mathcal{P}' \), shows that for all \( w \in V(Q) \),

\[
d(w, \mathcal{P}) = d(w, \mathcal{P}') \leq \frac{p - r + k + 1}{2}. \tag{24}
\]
Since for all \( w \in V(Q) \) and \( z \in V(R) \), \( wz \notin E(G) \), our degree condition yields
\[
d(z, \mathcal{P}') \geq \sigma_2(G) - d(w, Q) - d(w, \mathcal{P}') - d(z, R)
\]
\[
\geq (n + k) - (q - 1) - \frac{p - r + k + 1}{2} - (r - 1)
\]
\[
= \frac{p - r + k + 3}{2}.
\] (25)

If \( |R| = 1 \), since
\[
d(x^+, \mathcal{P}') \geq \frac{p - r + k + 3}{2} > \frac{p - r + k + 1}{2},
\]
there is a path \( P \in \mathcal{P}' \) and a vertex \( z \in V(P) \) such that \( x^+z, x^+z^+ \in E(G) \), so we can insert \( x^+ \) into \( P \), and obtain a larger \((A, B)\)-linkage than \( \mathcal{P} \).

If \( |R| = 2 \) then \( x^+ \neq y^- \), and (25) shows that
\[
d(x^+, \mathcal{P}') + d(y^-, \mathcal{P}') \geq p - r + k + 3.
\]
This implies that for some \( Z \in \mathcal{P}' \),
\[
d(x^+, P - R) + d(y^-, P - R) \geq |Z| + 1,
\] (26)

Note that we can choose \( Z \) to be of order at least 2 since for \( 2 \leq i \leq k \), \( |P_i| \geq 2 \), and if both \( P_1' \) and \( P_2'' \) have order 1, we still have
\[
d(x^+, P - P_1) + d(y^-, P - P_1) \geq |P - P_1| + 1.
\]
This shows that for some vertex \( z \in Z \) such that \( zx^+, z^+y^- \in E(G) \) or \( z^+x^+, zy^- \in E(G) \). Let us assume we are in the later case, since the other case is similar. If \( Z = P_i \) for some \( 2 \leq i \leq k \), replacing the path \( P_1 \) of \( \mathcal{P} \) with
\[
[a_1, x]_{P_1} \cup xu \cup S \cup vy \cup [y, b_1],
\]
and \( P_i \) with
\[
[a_i, z]_{P_i} \cup zy^- \cup R \cup x^+z^+ \cup [z^+, b_i]_{P_i},
\]
we contradict the maximality of \( \mathcal{P} \). If \( Z = P_1' \), we can replace the path \( P_1 \) of \( \mathcal{P} \) with
\[
[a_1, z]_{P_1} \cup R \cup x^+y^+ \cup [y^+, x]_{P_1} \cup xu \cup S \cup vy \cup [y, b_1]_{P_1},
\]
we again have a contradiction. The case $Z = P''_1$ is similar to the previous one.

To see that the condition $\sigma_2(G) \geq n + k - 1$ isn’t even enough to extend some $(A, B)$-systems to a Hamiltonian $(A, B)$-system, consider the graph $G_{n,k} = X + Y$ where $n \geq 4k$, $X$ is an empty graph (no edges) on $\frac{n-k+1}{2}$ vertices and $Y$ is a complete graph on $\frac{n+k-1}{2}$ vertices. Note that $G_{n,k}$ can be seen to be obtained by taking the complete graph $K_n$ and removing all vertices of a subgraph $X$ on $\frac{n-k+1}{2}$ vertices of $V(K_n)$.

One will verify that $\sigma_2(G_{n,k}) = n + k - 1$ yet if $A$ and $B$ are two disjoint $k$-sets of vertices of $X$, there can be no $(A, B)$-system in $G_{n,k}$ covering all the vertices of the graph.

References
