One or Two Disjoint Circuits Cover Independent Edges

Lovász–Woodall Conjecture

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In this paper, we prove the following theorem: Let \( L \) be a set of \( k \) independent edges in a \( k \)-connected graph \( G \). If \( k \) is even or \( G - L \) is connected, then there exist one or two disjoint circuits containing all the edges in \( L \). This theorem is the first step in the proof of the conjecture of L. Lovász (1974, Period. Math. Hungar., 82) and D. R. Woodall (1977, J. Combin. Theory Ser. B 22, 274–278). In addition, we give the outline of the proof of the conjecture and refer to the forthcoming papers.

Key Words: circuit; independent edges; Lovász conjecture.

1. INTRODUCTION

In this paper, all graphs considered are finite, undirected, and without loops or multiple edges. \( V(G) \) denotes the set of vertices of a given graph \( G \). A set of edges are disjoint, if no two of them have a vertex in common. A set of edges are independent edges, if any two of them are disjoint. In this paper, circuit \( C \) means a 2-regular connected subgraph. We often use the word “disjoint” as “vertex-disjoint.” Let \( k \)-cutset be a cutset consisting of \( k \) vertices. For a graph theoretic notation not defined here, we refer the reader to [1].

A well-known theorem of Dirac [3] states that given any \( k \) vertices in a \( k \)-connected graph \( G \), then \( G \) has a circuit containing all of them. He also proved that if \( e \) and \( f \) are two edges of \( k \)-connected graph, and if \( S \) is a set of \( k - 2 \) vertices of \( G \), then \( G \) contains a cycle which includes \( e, f \) and all the vertices in \( S \). Since then, many papers on this theme can be found in the literature: cf. Bondy and Lovász [2], Holton and Plummer [6], Holton et al. [7], Kaneko and Saito [8], and the author [9].
If \( L \) is a set of \( k \) independent edges in a \( k \)-connected graph \( G \) with \( k \) being odd, such that \( G - L \) is disconnected, then clearly \( G \) has no circuits containing all the edges of \( L \).

Considering this situation, Lovász [14] and Woodall [18] independently conjectured the following:

**Conjecture 1.** If \( k \) is even or \( G - L \) is connected, then \( G \) has a circuit containing all the edges of \( L \).

Conjecture 1 is well known to be true for \( k \leq 5 \). For \( k \leq 2 \), it is easily shown by using Menger’s Theorem. Lovász [15] proved the case of \( k = 3 \). Erdős and Győri [4] and Lomonosov [13] independently proved the case of \( k = 4 \). Sanders [16] proved the case of \( k = 5 \). Partial results concerning Conjecture 1 were due to Woodall [18] and Thomassen [17].

The final general result is proved by Häggkvist and Thomassen [5].

**Theorem 1.** If \( L \) is a set of \( k \) independent edges in a \((k+1)\)-connected graph \( G \), then there is a circuit containing all the edges in \( L \).

Note that Theorem 1 implies the conjecture of Berge [1, p. 214].

The purpose of this paper is to prove the following theorem.

**Theorem 2.** Let \( L \) be a set of \( k \) independent edges in a \( k \)-connected graph \( G \). If \( k \) is even or \( G - L \) is connected, then there exist one or two disjoint circuits containing all the edges in \( L \).

Note that the condition that \( k \) is even or \( G - L \) is connected is necessary as the same example of Conjecture 1 shows.

The proof involves a refinement of Woodall’s Hopping Lemma, which was introduced in [19] and applied in [5, 18, 19].

In Section 3, we outline the proof of Theorem 2 since this paper is long and technical.

Meanwhile, we prove Conjecture 1. In Section 5, we refer to our approach to Conjecture 1 and the forthcoming papers.

### 2. Preparation for the Proof of Theorem 2

Since the cases \( k \leq 3 \) were already proved, hence we may suppose \( k \geq 4 \). Assume that there do not exist one or two disjoint circuits containing all the edges in \( L \).

First of all, we prove the following lemma.

**Lemma 1.** There exists a path \( P \) which contains all the edges in \( L \).
Proof. Let $e$ be an edge in $L$. By using Theorem 1, we can get the fact that there exists a circuit $C$ containing $k-1$ edges in $L$. So we may assume that $C$ contains all the edges in $L\setminus e$. If $C$ contains $e$, then there exists a circuit containing all the edges in $L$. So, suppose that $C$ does not contain $e$.

Let $g$ and $h$ be the vertices of $e$. If $|V(e)\cap V(C)|=0$, since $k\geq 4$, there exists a path $P'$ connecting from $g$ to $C$. Then we can easily get the path containing all the edges in $L$ by using $e$, $P'$, and $C$.

If $|V(e)\cap V(C)|=1$, say $g\in V(C)$, then we can easily get the path containing all the edges in $L$. So, Lemma 1 follows.

Let $P$ be a path such that $P$ contains all the edges in $L$ and endvertices of $P$ are vertices that belong to $V(L)$. $P\setminus L$ consists of $k-1$ paths $P_1, \ldots, P_{k-1}$ and two endvertices of $P$. Let the vertices in order along $P_i$ be

$$x_{i,1}, x_{i,2}, \ldots, x_{i,m_i}$$

($i=1, \ldots, k-1$), where the edges $(x_{i,m_i}, x_{i+1,1})$ are edges in $L$. Let $a$ be the endvertex of $P$ adjacent to $x_{i,1}$ in $P$ and also, let $b$ be the endvertex of $P$ adjacent to $x_{k-1,m_{k-1}}$ in $P$.

Here is an extension of the definition of Woodall [18]. If $X\subseteq V(P)$ and if $X\cap P_i \neq \emptyset$, ($i=1, \ldots, k-1$), let $\inf_i(X)$ and $\sup_i(X)$ denote the following.

$$\inf_i(X) := x_{i,p}, \quad \text{where} \quad p := \inf\{q: x_{i,q} \in X\}$$

and

$$\sup_i(X) := x_{i,p}, \quad \text{where} \quad p := \sup\{q: x_{i,q} \in X\}.$$ 

For any $X\subseteq V(P)$, let $Fr_i(X)$, $Int_i(X)$ and $Cl_i(X)$ denote the following, respectively,

$$Fr_i(X) := \emptyset, \quad \text{if} \quad X\cap P_i = \emptyset$$

$$\{\inf_i(X), \sup_i(X)\} \quad \text{otherwise}$$

$$Int_i(X) := \emptyset, \quad \text{if} \quad |Fr_i(X)| \leq 1$$

$$x_{i,p}: \inf_i(X) < p < \sup_i(X) \quad \text{otherwise}$$

and

$$Cl_i(X) := Fr_i(X) \cup Int_i(X).$$
Let \( Fr(X), \) \( Int(X), \) and \( Cl(X) \) denote \( \bigcup_{i=1}^{k-1} Fr_i(X), \bigcup_{i=1}^{k-1} Int_i(X), \) and \( \bigcup_{i=1}^{k-1} Cl_i(X) \), respectively.

If \( H \) is a subgraph of \( G \) and if \( x \) and \( y \in V(G) \), \( x \ast y \) will always denote a path connecting \( x \) to \( y \) with \( (x \ast y) \cap P \subseteq \{x, y\} \). If \( X \subseteq V(G) \) and \( H \) is a subgraph of \( G \), let

\[
I(X, H) := \{ y \in V(P) : \text{there exists an } x \ast y \text{ in } G \setminus H, \text{ for some } x \in X \}.
\]

To the extension of Woodall’s definition [18], we define the sequence \( A_0 \leq A_1 \leq \cdots \) and the sequence \( B_0 \leq B_1 \leq \cdots \) of subsets of \( V(P) \), as

\[
A_0 := I(\{a\}, \{a\})
\]

\[
B_0 := I(\{b\}, \{b\})
\]

and, for any \( x, y \geq 1 \),

\[
A_x := A_{x-1} \cup I(\text{Int}(A_{x-1}), \{a\})
\]

\[
B_y := B_{y-1} \cup I(\text{Int}(B_{y-1}), \{b\}).
\]

\( A_{-1} \) and \( B_{-1} \) will be interpreted as \( \emptyset \). Note that there does not exist a path \( a \ast b \), for otherwise, there exists a circuit which contains all the edges in \( L \), which is contrary to the hypothesis.

Finally, if \( x \) and \( y \) are vertices occurring in order in a path \( P \), \( x, P, y \) and \( y, P, x \) will denote, respectively, the segment of \( P \) from \( x \) to \( y \), and the reverse segment from \( y \) to \( x \), and also if \( x \) and \( y \) are vertices occurring in order in a circuit \( C \), \( x, C, y \) and \( y, C, x \) will denote, respectively, the segment of \( C \) from \( x \) to \( y \), and the reverse segment from \( y \) to \( x \).

### 3. Outline of the Proof of Theorem 2

In this section, we give the outline of our proof. By Lemma 1, there exists a path \( P \) connecting \( a \) to \( b \). First, we prove the following;

(1) There do not exist distinct vertices \( a_x \) and \( b_y \) in \( P \), such that \( a_x \in A_i \) and \( b_y \in B_i \), for any \( x, y \geq 0 \) and for \( i = 1, \ldots, k-1 \).

Statement (1) is proved in Lemma 2. By (1), we have the following facts;

1. For any \( i \) with \( i = 1, \ldots, k-1 \), \( |Fr_i(A)| + |Fr_i(B)| \leq 2 \). Hence \( |Fr(A)| + |Fr(B)| \leq 2k - 2 \).
2. Since both \( \{x_{1,1}\} \cup Fr(A) \) and \( \{x_{k-1,m_k}\} \cup Fr(B) \) are cutsets, we can conclude that \( |Fr(A)| = |Fr(B)| = k - 1 \).
Next, we prove the following;

\(2\) \(A \cap B = \emptyset\).

Statement (2) is proved in Claim 2. By (1) and (2), since \(\frac{k-1}{2}\) is not an integer when \(k\) is even, either \(|Fr(A)| < k - 1\) or \(|Fr(B)| < k - 1\). Hence we have the following;

(3) We may assume that \(k\) is odd.

Since \(G-L\) is connected, there exists a path \(P'\) connecting from \(P_r\) to \(P_r\), where \(i' < i''\) and either \(A \cap V(P_r) \neq \emptyset\) and \(B \cap V(P_r) \neq \emptyset\) or \(B \cap V(P_r) \neq \emptyset\) and \(A \cap V(P_r) \neq \emptyset\). We can prove the following;

(4) \(B \cap V(P_r) = \emptyset\) and \(A \cap V(P_r) = \emptyset\).

We choose a path \(P'\) such that \(i'' - i'\) is as large as possible. Then we prove the following;

(5) \(i'' - i' \geq 2\).

Finally, we prove Claim 4, which immediately implies our theorem.

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4. PROOF OF THEOREM 2

We prove the following lemma.

**Lemma 2.** There do not exist distinct vertices \(a_x\) and \(b_y\) in \(P_i\) such that \(a_x \in A_x\) and \(b_y \in B_y\) for any \(x, y \geq 0\) and for \(i = 1, \ldots, k-1\).

**Proof.** If there exist such two distinct vertices \(a_x\) and \(b_y\) in \(P_i\), choosing \(x\) and \(y\) minimal, and considering two paths connecting \(a\) to \(b_y\) and \(a\) to \(b\), or \(a\) to \(a_x\) and \(b\) to \(b_y\) along one side of \(P_i\), we can consider the following statement which is extension of Woodall’s proof [18]:

\(X(x, y)\) There exist two disjoint paths \(R_{x, y}\) and \(R'_{x, y}\) such that one starts at \(a_x\) in \(A_x\) and terminates at \(b\) in \(B_y\) and the other starts at \(a\) and terminates at \(b\), or one starts at \(a_x\) in \(A_x\) and terminates at \(b\) and the other starts at \(a\) and terminates at \(b_y\), or one starts at \(a\) and terminates at \(a_x\) in \(A_x\) and the other starts at \(b\) and terminates at \(b_y\) in \(B_y\), such that the conditions (S1)–(S3) below are satisfied.

(S1) \(R_{x, y} \cup R'_{x, y}\) includes all the edges in \(L\) and all the vertices in \(\text{Int}(A_{x-1})\) and in \(\text{Int}(B_{y-1})\).
The only vertices of $R'_{x,y} \cup R_{x,y}$ not in $P$ occur in segment of $R_{x,y} \setminus L$ and $R'_{x,y} \setminus L$ of the form $w, w \ast r, r$, where $w$ and $r$ are both in $P$ but not both in $A_s$ or in $B_s$.

(S$_s$) For each of the paths $R_{x,y} \setminus L$ and $R'_{x,y} \setminus L$, say $Q_x$, and each $x' \leq x-1$ (or $y' \leq y-1$), if there is a vertex $q$ such that $a_x \ast q \cap (R_{x,y} \cup R'_{x,y}) \subseteq \{a_x, q\}$ (or $b_x \ast q \cap (R_{x,y} \cup R'_{x,y}) \subseteq \{b_x, q\}$), where $q \in Q_x \cap \text{Int}(A_s)$ (or $q \in Q_x \cap \text{Int}(B_s)$), then there are two vertices of $\text{Fr}(A_s)$ (or $\text{Fr}(B_s)$) occurring before and after $q$ along $Q_x$, and each of the vertices between them along $Q_x$ is in $\text{Int}(A_s)$ (or in $\text{Int}(B_s)$).

To prove Lemma 2, it is sufficient to prove the following claim.

CLAIM 1. If $X(x, y)$ holds, then $G$ has one or two disjoint circuits containing all the edges in $L$.

Proof. We prove Claim 1 by the induction on $x + y$. Suppose $x + y = 0$. Let $T_a$ be a path from $a$ to $a_0$, and let $T_b$ be a path from $b$ to $b_0$. If $T_a$ and $T_b$ are disjoint, by the definition of the set $A_0$ and $B_0$, we can get the result that $G$ has one circuit that contains all the edges in $L$ or $G$ has two disjoint circuits that contain all the edges in $L$.

So we can suppose that $T_a \cap T_b \neq \emptyset$. But in this case, since there exists a path $a \ast b$, the result easily follows.

Suppose $x + y > 0$. Without loss of generality, we may assume $x > 0$. If $a_x \in A_{x-1}$, then, the result follows by the induction hypothesis. So, we may assume $a_x \in A_s \setminus A_{x-1}$. Let $Q_x$ be the path $R_{x,y} \setminus L$ or $R_{x',y} \setminus L$ such that $Q_x$ contains $a_x$. Then, we can choose a path $a_x \ast y_{x-1}$ connecting $a_x$ to $y_{x-1}$, where $y_{x-1} \in \text{Int}(A_{x-1})$. This path does not intersect any segment $w, w \ast r, r$ in $R_{x,y}$ or in $R'_{x,y}$ with $w$ and $r$ in $P$ except for its end vertices $a_x$ and $y_{x-1}$.

For otherwise, both $w$ and $r$ are in $A_s$, which is contrary to (S$_s$). By the condition (S$_s$), there exists a vertex $a_{x-1} \in A_{x-1}$ which is preceding $y_{x-1}$ such that the segment $a_{x-1}, R_{x,y}, y_{x-1}$ or $a_{x-1}, R'_{x,y}, y_{x-1}$ does not contain edges in $L$. Now we choose a vertex $a'_x$ which is the last vertex before $y_{x-1}$ along $R_{x,y}$ (if $y_{x-1} \in R_{x,y}$) or along $R'_{x,y}$ (if $y_{x-1} \in R'_{x,y}$) and $a'_x$ is in $\text{Fr}(A_s)$ for any $x' \leq x - 1$, and choose $x'$ minimal so that $a'_x \notin \text{Cl}(A_{x-1})$. Also, by the condition (S$_s$), there exists a vertex $a'_{x-1} \in A_{x-1}$ which is succeeding $y_{x-1}$ such that the segment $y_{x-1}, R_{x,y}, a'_{x-1}$ or $y_{x-1}, R'_{x,y}, a'_{x-1}$ does not contain edges in $L$. Now we choose a vertex $a'_x$ which is the last vertex after $y_{x-1}$ along $R_{x,y}$ (if $y_{x-1} \in R_{x,y}$) or along $R'_{x,y}$ (if $y_{x-1} \in R'_{x,y}$) and $a'_x$ is in $\text{Fr}(A_s)$ for any $x' \leq x - 1$, and choose $x'$ minimal so that $a'_x \notin \text{Cl}(A_{x-1})$. We will write $a_x$ instead of $a'_x$ since it may not be confusing for readers.

Then there does not exist a vertex that is in $\text{Int}(A_{x-1})$ in the segments both $a_x, R_{x,y}, y_{x-1}$ and $y_{x-1}, R_{x,y}, a_x$ (if $y_{x-1} \in R_{x,y}$), or both $a_x, R'_{x,y}, y_{x-1}$, and $y_{x-1}, R'_{x,y}, a_x$ (if $y_{x-1} \in R'_{x,y}$).
follows by the induction hypothesis. Hence Claim 1 follows.

otherwise, there must exist some $P_i$ of $P$ which contains distinct vertices $r$, $w$ such that $r \in A_{x−1}$ and $w \in B_{y−1}$. But, in this case, choosing the paths connecting $a$ to $r$ along $P$ and $b$ to $w$ along $P$, or $a$ to $w$ along $P$ and $b$ to $r$ along $P$, there exist two disjoint paths $R'_{x−1,y−1}$ and $R'_{x−1,y−1}$, and hence, the result follows by the induction hypothesis.

We consider three cases for $R_{x,y}$ and $R'_{x,y}$.

**Case 1.** $R_{x,y}$ is a path which starts at $a_x$ in $A_x$ and terminates at $b_y$ in $B_y$. $R'_{x,y}$ is a path which starts at $a$ and terminates at $b$.

In this case, if $y_{x−1} \in R_{x,y}$, then we can replace the path $R'_{x,y}$ such that $a_x, R_{x−1,y}, a_{x−1}, y_{x−1}, R_{y−1}, b$. And $R'_{x,y}$ is $R'_{x,y}$. These two paths satisfy the case of $x'+y$. So, the result follows by the induction hypothesis.

If $y_{x−1} \in R'_{x,y}$, then we can replace the path $R'_{x,y}$ such that $b, R_{x−1,y}, y_{x−1}, R_{y−1}, a_x, a_{x−1}, R'_{x,y}, b$. And also we can replace the path $R'_{x,y}$ such that $a, R'_{x,y}, a_{x−1}, y_{x−1}, R_{{x−1},y}, b$. These two paths satisfy the case of $x'+y$. So, the result follows by the induction hypothesis.

**Case 2.** $R_{x,y}$ is a path which starts at $a_x$ in $A_x$ and terminates at $b$ and $R'_{x,y}$ is a path which starts at $a$ and terminates at $b_y$ in $B_y$.

In this case, if $y_{x−1} \in R_{x,y}$, then we can replace the path $R'_{x,y}$ such that $a_x, R_{x−1}, y_{x−1}, R_{y−1}, b$. And $R'_{x,y}$ is $R'_{x,y}$. These two paths satisfy the case of $x'+y$. So, the result follows by the induction hypothesis.

If $y_{x−1} \in R'_{x,y}$, then we can replace the path $R'_{x,y}$ such that $b, R_{x−1}, a_x, a_{x−1}, y_{x−1}, R_{{x−1},y}, b$. And also we can replace the path $R'_{x,y}$ such that $a, R'_{x,y}, a_{x−1}, y_{x−1}, R_{{x−1},y}, b$). These two paths satisfy the case of $x'+y$. So, the result follows by the induction hypothesis.

**Case 3.** $R_{x,y}$ is a path which starts at $a$ and terminates at $a_x$ in $A_x$ and $R'_{x,y}$ is a path which starts at $b$ and terminates at $b_y$ in $B_y$.

In this case, if $y_{x−1} \in R_{x,y}$, then we can replace the path $R'_{x,y}$ such that $a, R_{x−1}, y_{x−1}, R_{y−1}, a_x, a_{x−1}, R'_{x,y}$. These two paths satisfy the case of $x'+y$. So, the result follows by the induction hypothesis.

If $y_{x−1} \in R'_{x,y}$, then we can replace the path $R'_{x,y}$ such that $a, R_{x−1}, a_x, a_{x−1}, y_{x−1}, R'_{x,y}, b$. And also we can replace the path $R'_{x,y}$ such that $a, R'_{x,y}, a_{x−1}, y_{x−1}, R_{{x−1},y}, b$). These two paths satisfy the case of $x'+y$. So, the result follows by the induction hypothesis. Hence Claim 1 follows.

Consequently, Lemma 2 follows.

We shall remark the proof of Claim 1. If $R_{x,y}$ and $R'_{x,y}$ satisfy either Case 1 or Case 2, by using the argument in the proof of Claim 1, we know
that we can get $R_{x', j}$ and $R'_{x', j}$ which also satisfy either Case 1 or Case 2. This will be used in the remaining part of our proof, in particular, in the proof of Claim 4.

Now we can suppose that there are no two distinct vertices $a_x$ and $b_y$ such that $a_x \in A_x$ and $b_y \in B_y$ in any segments of $P_i$.

Since $V(P)$ is finite, the sequence of sets $A_0 \subseteq A_1 \subseteq \cdots$ and the sequence $B_0 \subseteq B_1 \subseteq \cdots$ must be constant from some point onwards. Let $A$ and $B$ be the final sets.

We can easily get the fact that for each $i (1 \leq i \leq k-1)$,

$$|\text{Fr}_i(A)| + |\text{Fr}_i(B)| \leq 2$$

and since $P \setminus (L \cup \{a\} \cup \{b\})$ has $k-1$ segments, we have

$$|\text{Fr}(A)| + |\text{Fr}(B)| \leq 2k-2.$$

So,

$$|\text{Fr}(A)| \leq k-1$$

and

$$|\text{Fr}(B)| \leq k-1.$$ 

Hence we can get the following:

$$|\text{Fr}(A)| = |\text{Fr}(B)| = k-1.$$ 

For otherwise, $\text{Fr}(A) \cup \{x_{i-1}\}$ or $\text{Fr}(B) \cup \{x_{k-1,m-1}\}$ is a cutset separating $a$ from $b$, and its cardinality is at most $k-1$, which is contrary to the connectivity of $G$.

Hence, we may assume, for any $P_i$,

$$|\text{Fr}_i(A)| + |\text{Fr}_i(B)| = 2.$$ 

We prove the following claim.

**Claim 2.** $A \cap B = \emptyset$.

**Proof.** Assume, to the contrary. Let $x_{i,j} \in A \cap B$. First, we prove the following subclaims.

**Subclaim 1.** If there exist two paths $l_1$ and $l_2$, and a cycle $C_1$, such that $l_1$ is connecting from $a$ to $b$, and $l_2$ is connecting form $a_x$ to $b_y$, or $l_1$ is connecting from $a_x$ to $b_y$, and $l_2$ is connecting form $a$ to $b$, and also $l_1 \cup l_2 \cup C_1$ satisfies the following conditions.
(S_1) \( I_1 \cup I_2 \cup C_1 \) includes all the edges in \( L \) and all the vertices in \( \text{Int}(A_{x-1}) \) and in \( \text{Int}(B_{y-1}) \).

(S_2) The only vertices of \( I_1 \cup I_2 \cup C_1 \) not in \( P \) occur in segment of \( I_1 \setminus L, I_2 \setminus L \) and \( C_1 \setminus L \) of the form \( w, w \ast r, r \), where \( w \) and \( r \) are both in \( P \) but not both in \( A_x \) or in \( B_y \).

(S_3) For each of the paths of \( I_1 \setminus L, I_2 \setminus L \) and \( C_1 \setminus L \), say \( Q_i \), and each \( x' \leq x-1 \) (or \( y' \leq y-1 \)), if there is a vertex \( q \) such that \( q \in Q_i \cap \text{Int}(A_x) \) (or \( q \in Q_i \cap \text{Int}(B_y) \)) then there are two vertices of \( \text{Fr}(A_x) \) (or \( \text{Fr}(B_y) \)) occurring before and after \( q \) along \( Q_i \), and each of the vertices between them along \( Q_i \) is in \( \text{Int}(A_x) \) (or in \( \text{Int}(B_y) \)).

Then there exist one or two disjoint circuits which contain all the edges in \( L \).

**Proof.** We prove Subclaim 1 by the induction on \( x+y \). Suppose that \( x+y=0 \).

Let \( T_y \) be a path from \( a \) to \( a_0 \) and let \( T_y \) be a path from \( b \) to \( b_0 \). If \( T_y \) and \( T_y \) are disjoint, by the definition of the set \( A_0 \) and \( B_0 \), we can get the result that \( G \) has two disjoint circuits that contain all the edges in \( L \).

So we can suppose that \( T_y \cap T_y \neq \emptyset \). But in this case, since there exists a path \( a \ast b \), the result easily follows.

Suppose \( x+y > 0 \). Without loss of generality, we may assume \( x > 0 \). If \( a_0 \in A_{x-1} \), then the result follows by the induction hypothesis. So, \( a_0 \in A_x \setminus A_{x-1} \). We can choose a path \( a_0 \ast y_{x-1} \) connecting \( a_0 \) to \( y_{x-1} \), where \( y_{x-1} \in \text{Int}(A_{x-1}) \). This path does not intersect any segment \( w, w \ast r, r \) in \( I_1 \) or in \( I_2 \) or in \( C_1 \) with \( w \) and \( r \) in \( P \) except for its end vertices \( a_0 \) and \( y_{x-1} \). For otherwise, both \( w \) and \( r \) are in \( A_x \), which is contrary to (S_2). By the condition (S_1), there exists a vertex \( a_{x-1} \in A_{x-1} \) which is preceding \( y_{x-1} \), such that the segment \( a_{x-1}, l_1, y_{x-1} \) or \( a_{x-1}, l_2, y_{x-1} \) or \( a_{x-1}, C_1, y_{x-1} \) does not contain edges in \( L \). We choose a vertex \( a_{x'} \) which is the last vertex before \( y_{x-1} \) along \( l_1 \) (if \( y_{x-1} \in l_1 \)) or along \( l_2 \) (if \( y_{x-1} \in l_2 \)) or along \( C_1 \) (if \( y_{x-1} \in C_1 \)) and \( a_{x'} \) is in \( \text{Fr}(A_x) \) for any \( x' \leq x-1 \), and choose \( x' \) minimal so that \( a_{x'} \notin \text{Cl}(A_{x-1}) \).

Also, by the condition (S_2), there exists a vertex \( a_{x-1}' \in A_{x-1} \) which is succeeding \( y_{x-1} \) such that the segment \( y_{x-1}, l_1, a_{x-1}', y_{x-1}, l_2, a_{x-1}', y_{x-1}, C_1, a_{x-1}' \) does not contain edges in \( L \). We choose a vertex \( a_{x'}' \) which is the last vertex after \( y_{x-1} \) along \( l_1 \) (if \( y_{x-1} \in l_1 \)) or along \( l_2 \) (if \( y_{x-1} \in l_2 \)) or along \( C_1 \) (if \( y_{x-1} \in C_1 \)) and \( a_{x'}' \) is in \( \text{Fr}(A_x) \) for any \( x' \leq x-1 \), and choose \( x' \) minimal so that \( a_{x'}' \notin \text{Cl}(A_{x-1}) \). We will write \( a_{x'} \) instead of \( a_{x'}' \) since it may not be confusing for readers.

Then there does not exist a vertex that is in \( \text{Int}(A_{x-1}) \) in the segments both \( a_{x'}, l_1, y_{x-1} \) and \( y_{x-1}, l_1, a_{x} \) (if \( y_{x-1} \in l_1 \)), or both \( a_{x'}, l_2, y_{x-1} \) and \( y_{x-1}, l_2, a_{x} \) (if \( y_{x-1} \in l_2 \)), or both \( a_{x'}, C_1, y_{x-1} \) and \( y_{x-1}, C_1, a_{x} \) (if
\( y_{x-1} \in C_1 \). We may assume that there are no vertices in \( \text{Int}(B_r) \) in the segments both \( a_x, l_1, y_{x-1} \) and \( y_{x-1}, l_1, a_x \) (if \( y_{x-1} \in l_1 \)), or both \( a_x, l_2, y_{x-1} \) and \( y_{x-1}, l_2, a_x \) (if \( y_{x-1} \in l_2 \)). For otherwise, there must exist some \( P_i \) of \( P \) which contains distinct vertices \( r \) and \( w \) such that \( r \in A_{x-1} \) and \( w \in B_{x-1} \). But, in this case, choosing the paths connecting \( a \) to \( r \) along \( P \) and \( b \) to \( w \) along \( P \), or \( a \) to \( w \) along \( P \) and \( b \) to \( r \) along \( P \), there exist \( l_1 \) and \( l_2 \) which satisfy Claim 1, and hence, the result follows. We consider two cases for \( l_1 \) and \( l_2 \).

**Case 1.** \( l_1 \) is connecting from \( a \) to \( b \) and \( l_2 \) is connection form \( a_x \) to \( b_y \).

In this case, if \( y_{x-1} \in l_2 \), then we can replace the path \( l_2 \) such that \( a_x, l_2, a_x \). \( \bar{a}_x, a_x \) and \( y_{x-1}, l_2, b, l_2 \). These two paths \( l_1 \) and \( l_2 \), and a cycle \( C_1 \) satisfy the case of \( x' + y \). So, the result follows by the induction hypothesis.

If \( y_{x-1} \in l_1 \), then we can replace the path \( l_2 \) such that \( a_x, l_2, y_{x-1}, y_{x-1}, a_x, l_2, b, l_2 \) is still \( l_1 \). These two paths \( l_1 \) and \( l_2 \), and a cycle \( C_1 \) satisfy the case of \( x' + y \). So, the result follows by the induction hypothesis.

If \( y_{x-1} \in C_1 \), then we can get two paths \( a_x, C_1, y_{x-1}, y_{x-1}, a_x, l_2, b \) and \( l_1 \), which satisfy Claim 1, and hence, the result follows.

**Case 2.** \( l_1 \) is connecting from \( a \) to \( b \) and \( l_2 \) is connecting form \( a_x \) to \( b_y \).

In this case, if \( y_{x-1} \in l_2 \), then we can replace the path \( l_2 \) such that \( a_x, l_2, a_x \). \( \bar{a}_x, a_x \) and \( y_{x-1}, l_2, b, l_2 \). These two paths \( l_1 \) and \( l_2 \), and a cycle \( C_1 \) satisfy the case of \( x' + y \). So, the result follows by the induction hypothesis.

If \( y_{x-1} \in l_1 \), then we can replace the path \( l_2 \) such that \( a_x, l_2, b \) and also we can replace the path \( l_1 \) such that \( a, l_1, y_{x-1}, y_{x-1}, a_x, l_2, b, l_1 \) is still \( l_1 \). These two paths \( l_1 \) and \( l_2 \), and a cycle \( C_1 \) satisfy the case of \( x' + y \). So, the result follows by the induction hypothesis.

If \( y_{x-1} \in C_1 \), then we can get two paths \( l_1 \) and \( a_x, C_1, y_{x-1}, y_{x-1}, a_x, l_2, b \) which satisfy Claim 1, and hence, the result follows.

**Subclaim 2.** For any vertex \( v \in V(P_1), v \notin A \), and for any vertex \( u \in V(P_{k-1}), u \notin B \).

**Proof.** Suppose that there exists a vertex \( v \in V(P_1) \) such that \( v \in A \). If \(|\text{Fr}_1(A)| = 1 \) and \( v = v_{1,1} \), then, since

\[
|\text{Fr}(A)| = |\text{Fr}(B)| = k - 1, 
\]

\( \text{Fr}(A) \) is a cutset separating \( a \) from \( b \) and its cardinality is \( k - 1 \), which is contrary to the connectivity of \( G \). So, we may assume that \( sup_i(A) \)
is not \( x_{1,1} \). Let \( x_{1,s} \) be \( \text{sup}_1(A) \). Also, let \( Q \) be the set of vertices in \( x_{1,1}, P_1, x_{1,s-1} \). Let \( C_0 \) be

\[
C_0 := I(Q, Q) \cup \{x_{1,s}\} \cup Q
\]

and for \( i \geq 1 \), let \( C_i \) be

\[
C_i := C_{i-1} \cup I(\text{Int}(C_{i-1}), Q).
\]

\( C_{-1} \) will be interpreted as \( \emptyset \). We prove the following statement: There do not exist distinct vertices \( c_x \) and \( b_y \) in \( P_i \) such that \( c_x \in C_x \) and \( b_y \in B_y \) for any \( x \geq 0 \), for some \( y \geq 0 \) and for \( i = 1, \ldots, k-1 \).

**Proof.** If there exist such two distinct vertices \( c_x \) and \( b_y \) in \( P_i \), choosing \( x \) minimal, and considering two paths connecting \( a \) to \( b_y \) and \( c_x \) to \( b_y \), or \( a \) to \( c_x \) and \( b_y \) to \( b \) along one side of \( P \), we can consider the following statement:

\( X(x) \) There exist two disjoint paths \( R_x \) and \( R'_x \) that one starts at \( c_x \) in \( C_x \) and terminates at \( b_y \) in \( B_y \) and the other starts at \( a \) and terminates at \( b_y \), or one starts at \( c_x \) in \( C_x \) and terminates at \( b_y \) and the other starts at \( a \) and terminates at \( b_y \) in \( B_y \), or one starts at \( c_x \) in \( C_x \) and terminates at \( a \) and the other starts at \( b_y \) and terminates at \( b_y \) in \( B_y \), such that conditions \((S_1)-(S_3)\) below are satisfied.

\[(S_1)\] \( R_x \cup R'_x \) includes all the edges in \( L \) and all the vertices in \( \text{Int}(C_{x-1}) \), \( \text{Int}(A) \), \( \text{Int}(B) \) and \( Q \) for \( x \geq 1 \).

\[(S_2)\] For any \( c_0 \in C_0 \), if there exists a path \( c_0 \ast q \) where \( q \in Q \), then \( c_0 \ast q \cap \{R_x \cup R'_x\} \subseteq \{q, c_0\} \), and the only vertices of \( R_x \cup R'_x \) not in \( P \) occur in segment of \( R_x \setminus L \) and \( R'_x \setminus L \) of the form \( w, w \ast r, r \), where \( w \) and \( r \) are both in \( P \) but not both in \( C_x \).

\[(S_3)\] For each of the paths \( R_x \setminus L \) and \( R'_x \setminus L \), say \( S_x \), and each \( x' \leq x-1 \), if there is a vertex \( s \) such that \( s \in S_x \cap \text{Int}(C_{x'}) \), then there are two vertices of \( \text{Fr}(C_{x'}) \) occurring before and after \( s \) along \( S_x \) and each of the vertices between them along \( S_x \) is in \( \text{Int}(C_{x'}) \) and if \( S_x \) contains a vertex \( x_{1,s} \), then \( x_{1,s} \) is adjacent to \( Q \) in \( S_x \) and furthermore \( S_x \) contains the segment \( x_{1,1}, P_1, x_{1,s-1} \).

It is sufficient to prove the following statement.

*If \( X(x) \) holds, then \( G \) has one or two disjoint circuits containing all the edges in \( L \).*

**Proof.** We prove by induction on \( x \). Suppose \( x = 0 \).

Let \( T_0 \) be a path \( T_0 = q \ast c_0 \), where \( q \in Q \). By the condition \((S_2)\), \( q \ast c_0 \cap \{R_0 \cup R'_0\} \subseteq \{q, c_0\} \). Note that \( x_{1,s} \in A_{x'} \) for some \( x' \). Also, by the
condition (S₁), either \( q, R_0, c_0 \) or \( c_0, R_0, q \) or \( q, R'_0, c_0 \) or \( c_0, R'_0, q \) does not contain any vertices in \( \text{Int}(A) \cup \text{Int}(B) \).

We consider three cases for \( R_0 \) and \( R'_0 \).

**Case 1.** \( R_0 \) is a path which starts at \( c_0 \) in \( C_0 \) and terminates at \( b \) in \( B_0 \), and \( R'_0 \) is a path which starts at \( a \) and terminates at \( b \).

Suppose \( q \in R_0 \). If \( c_0, R_0, x_{1,z} \) is shorter than \( c_0, R_0, q \), then we can get two paths \( x_{1,z}, \overrightarrow{R_0}, c_0 \cdot q, q, R_0, b \) and \( R'_0 \) which satisfy Claim 1, and hence, the result follows. If \( c_0, R_0, x_{1,z} \) is longer than \( c_0, R_0, q \), then we can get two paths \( x_{1,z}, R_0, b \) and \( R'_0 \), and a cycle \( c_0, R_0, q, q \cdot c_0, c_0 \) which satisfy Subclaim 1, and hence, the result follows.

Suppose \( q \in R'_0 \). If \( a, R'_0, x_{1,z} \) is shorter than \( a, R'_0, q \), then we can get two paths \( a, R'_0, x_{1,z} \) and \( b, \overrightarrow{R'_0}, q, q \cdot c_0, c_0, R_0, b \), which satisfy Claim 1, and hence, the result follows. If \( a, R'_0, x_{1,z} \) is longer than \( a, R'_0, q \), then we can get two paths \( x_{1,z}, R'_0, b \) and \( a, R'_0, q, q \cdot c_0, c_0, R_0, b \), which satisfy Claim 1, and hence, the result follows.

**Case 2.** \( R_0 \) is a path which starts at \( c_0 \) in \( C_0 \) and terminates at \( b \) and \( R'_0 \) is a path which starts at \( a \) and terminates at \( b \) in \( B_0 \).

Suppose \( q \in R_0 \). If \( c_0, R_0, x_{1,z} \) is shorter than \( c_0, R_0, q \), then we can get two paths \( x_{1,z}, \overrightarrow{R_0}, c_0 \cdot q, q, R_0, b \) and \( R'_0 \) which satisfy Claim 1, and hence, the result follows. If \( c_0, R_0, x_{1,z} \) is longer than \( c_0, R_0, q \), then we can get two paths \( x_{1,z}, R_0, b \) and \( R'_0 \), and a cycle \( c_0, R_0, q, q \cdot c_0, c_0 \) which satisfy Subclaim 1, and hence, the result follows.

Suppose \( q \in R'_0 \). If \( a, R'_0, x_{1,z} \) is shorter than \( a, R'_0, q \), then we can get two paths \( a, R'_0, x_{1,z} \) and \( b, \overrightarrow{R'_0}, c_0 \cdot q, q, R'_0, b \), which satisfy Claim 1, and hence, the result follows. If \( a, R'_0, x_{1,z} \) is longer than \( a, R'_0, q \), then we can get two paths \( x_{1,z}, R'_0, b \) and \( a, R'_0, q, q \cdot c_0, c_0, R_0, b \) which satisfy Claim 1, and hence, the result follows.

**Case 3.** \( R_0 \) is a path which starts at \( c_0 \) in \( C_0 \) and terminates at \( a \) and \( R'_0 \) is a path which starts at \( b \) and terminates at \( b \) in \( B_0 \).

Suppose \( q \in R_0 \). If \( c_0, R_0, x_{1,z} \) is shorter than \( c_0, R_0, q \), then we can get two paths \( x_{1,z}, \overrightarrow{R_0}, c_0 \cdot q, q, R_0, a \) and \( R'_0 \) which satisfy Claim 1, and hence, the result follows. If \( c_0, R_0, x_{1,z} \) is longer than \( c_0, R_0, q \), then we can get two paths \( x_{1,z}, R_0, a \) and \( R'_0 \), and a cycle \( c_0, R_0, q, q \cdot c_0, c_0 \) which satisfy Subclaim 1, and hence, the result follows.

Suppose \( q \in R'_0 \). If \( b, R'_0, x_{1,z} \) is shorter than \( b, R'_0, q \), then we can get two paths \( b, R'_0, x_{1,z} \) and \( b, \overrightarrow{R'_0}, q, q \cdot c_0, c_0, R'_0, a \) which satisfy Claim 1, so, the result follows. If \( a, R'_0, x_{1,z} \) is longer than \( a, R'_0, q \), then we can get two paths \( x_{1,z}, R'_0, b \) and \( b, R'_0, q, q \cdot c_0, c_0, R_0, a \) which satisfy Claim 1, and hence, the result follows.

Suppose \( x > 0 \). If \( c_e \in C_{x−1} \), the result follows by the induction hypothesis. So, we may assume \( c_e \in C_0 \setminus C_{x−1} \). We can choose a path \( c_e \cdot y_{x−1} \)
connecting $c_x$ to $y_{x-1}$, where $y_{x-1} \in \text{Int}(C_{x-1})$. This path does not intersect any segment $w$, $w \cdot r$, $r$ in $R_x$ or in $R'_x$ with $w$ and $r$ in $P$ except for its end vertices $c_x$ and $y_{x-1}$. For otherwise, both $w$ and $r$ are in $C_{x-1}$, which is contrary to $(S_i)$. Note that $y_{x-1} \notin \text{Int}(A)$, for otherwise, we can choose two vertex disjoint paths which satisfy Claim 1, a contradiction.

By the condition $(S_i)$, there exists a vertex $c_{x-1} \in C_{x-1}$ which is preceding $y_{x-1}$ such that the segment $c_{x-1}$, $R_x$, $y_{x-1}$ or $c_{x-1}$, $R'_x$, $y_{x-1}$ does not contain edges in $L$. Now we choose a vertex $c_y$ which is the last vertex before $y_{x-1}$ along $R_x$ (if $y_{x-1} \in R_x$) or along $R'_x$ (if $y_{x-1} \in R'_x$) and $c_y$ is in $\text{Fr}(C_x)$ for any $x' \leq x-1$, and choose $x'$ minimal so that $c_y \notin \text{Cl}(C_{x-1})$. Also, by the condition $(S_i)$, there exists a vertex $c'_{x-1} \in C_{x-1}$ which is succeeding $y_{x-1}$ such that the segment $y_{x-1}$, $R_x$, $c'_{x-1}$ or $y_{x-1}$, $R'_x$, $c'_{x-1}$ does not contain edges in $L$. Now we choose a vertex $c'_{y}$ which is the last vertex after $y_{x-1}$ along $R_x$ (if $y_{x-1} \in R_x$) or along $R'_x$ (if $y_{x-1} \in R'_x$) and $c_y$ is in $\text{Fr}(C_x)$ for any $x' \leq x-1$, and choose $x'$ minimal so that $c'_{y} \notin \text{Cl}(C_{x-1})$. We will write $c_y$ instead of $c'_{y}$, since it may not be confusing for readers.

Then there does not exist a vertex that is in $\text{Int}(C_{x-1})$ in the segments both $c_y$, $R_x$, $y_{x-1}$ and $y_{x-1}$, $R_x$, $c_y$ (if $y_{x-1} \in R_x$), or both $c_y$, $R'_x$, $y_{x-1}$ and $y_{x-1}$, $R'_x$, $c_y$ (if $y_{x-1} \in R'_x$). We may assume that there are no vertices in $\text{Int}(B)$ in the segments both $c_y$, $R_x$, $y_{x-1}$ and $y_{x-1}$, $R_x$, $c_y$ (if $y_{x-1} \in R_x$), or both $c_y$, $R'_x$, $y_{x-1}$ and $y_{x-1}$, $R'_x$, $c_y$ (if $y_{x-1} \in R'_x$). For otherwise, there exist some $P_i$ of $P$ which contains distinct vertices $r, w$ such that $r \in C_{x-1}$ and $w \in B$. But, in this case, choosing the paths connecting $a$ to $r$ and $b$ to $w$ along $P_i$, or $a$ to $w$ and $b$ to $r$ along $P_i$ there exist two disjoint paths $R_{x-1}$ and $R'_{x-1}$, and hence the result follows by the induction hypothesis. If there exists a vertex in $\text{Int}(A)$ in the segments $c_y$, $R_x$, $y_{x-1}$ or $y_{x-1}$, $R_x$, $c_y$ (if $y_{x-1} \in R_x$), or $c_y$, $R'_x$, $y_{x-1}$ or $y_{x-1}$, $R'_x$, $c_y$ (if $y_{x-1} \in R'_x$), then we choose vertices $a_y$ which are either the last vertex before $y_{x-1}$ along $R_x$ or the last vertex after $y_{x-1}$ along $R_x$ (if $y_{x-1} \in R_x$), or the last vertex before $y_{x-1}$, $R'_x$ or the last vertex after $y_{x-1}$ along $R'_x$ (if $y_{x-1} \in R'_x$), and $a_y$ is in $\text{Fr}(A)$ for any $x'' \leq x'-1$, and choose $x''$ minimal so that $a_y \notin \text{Cl}(A_{x-1})$. Then we assume $a_y$ as $c_y$.

Now we consider three cases for $R_x$ and $R'_x$.

**Case 1.** $R_x$ is a path which starts at $c_x$ in $C_x$ and terminates at $b_y$ in $B_x$ and $R'_x$ is a path which starts at $a$ and terminates at $b$.

First, assume $c_y$ is not $a_y$. If $y_{x-1} \in R_x$, then we can replace the path $R_x$ such that $c_x$, $R_x$, $c_y$ and $y_{x-1}$. If $y_{x-1} \in R'_x$, then we can replace the path $R'_x$ such that $b_y$, $R'_x$. These two paths satisfy the case of $x'$. So, the result follows by the induction hypothesis.

If $y_{x-1} \in R_x$, then we can replace the path $R_x$ such that $b$, $R_x$, $y_{x-1}$, $c_x$, $R_x$, $b_y$, and also we can replace the path $R'_x$ such that $c_y$, $R'_x$, $a$. These two paths satisfy the case of $x'$. So, the result follows by the induction hypothesis.
Finally, suppose \( c' \) is \( a' \). Then by the same way, we can get two disjoint paths which satisfy Claim 1. Hence the result follows.

**Case 2.** \( R' \) is a path which starts at \( c' \) in \( C' \) and terminates at \( b' \) and \( R'_c \) is a path which starts at \( a' \) and terminates at \( b' \) in \( B' \).

First, assume \( c' \) is not \( a' \). If \( y_{x-1} \in R'_c \), then we can replace the path \( R'_c \) such that \( c'_x, R'_c, c_x, y_{x-1}, y_{x-1}, R' \), \( b' \). And \( R'_c \) is \( R'_c' \). These two paths satisfy the case of \( x' \). So, the result follows by the induction hypothesis.

If \( y_{x-1} \in R'_c \), then we can replace the path \( R'_c' \) such that \( b', \overline{R'_c}, c_x, c'_x, y_{x-1}, y_{x-1}, R'_c \), \( b' \), and also we can replace the path \( R'_c \) such that \( c'_x, R'_c', a \). These two paths satisfy the case of \( x' \). So, the result follows by the induction hypothesis.

Finally, suppose \( c' \) is \( a' \). Then by the same way, we can get two disjoint paths which satisfy Claim 1. Hence the result follows.

**Case 3.** \( R'_c \) is a path which starts at \( c'_x \) in \( C'_c \) and terminates at \( a' \) and \( R'_c' \) is a path which starts at \( b' \) and terminates at \( b'_y \) in \( B' \).

First, assume \( c'_x \) is not \( a' \). If \( y_{x-1} \in R'_c \), then we can replace the path \( R'_c \) such that \( c'_x, R'_c, c_x, c'_x, y_{x-1}, y_{x-1}, R_c \), \( a \). And \( R'_c \) is \( R'_c' \). These two paths satisfy the case of \( x' \). So, the result follows by the induction hypothesis.

If \( y_{x-1} \in R'_c' \), then we can replace the path \( R'_c' \) such that \( a, \overline{R'_c}, c_x, c'_x, y_{x-1}, y_{x-1}, R'_c' \), \( b' \), and also we can replace the path \( R'_c \) such that \( c'_x, R'_c', b'_y \). These two paths satisfy the case of \( x' \). So, the result follows by the induction hypothesis.

Finally, suppose \( c'_x \) is \( a' \). Then by the same way, we can get two disjoint paths which satisfy Claim 1. Hence the result follows.

Since \( V(P) \) is finite, the sequence of sets \( C_0 \subseteq C_1 \subseteq \cdots \) must be constant from some point onwards. Let \( C \) be the final sets. As \( |Fr(A)| = |Fr(B)| = k-1 \), \( |Fr(C)| = k-1 \).

Then, \( (Fr(C) \setminus \{x_1, 1\}) \cup \{a\} \) is a cutset separating \( Q \) from \( b \) and its cardinality is at most \( k - 1 \), which is contrary to the connectivity of \( G \).

The case of \( P_{x-1} \) follows by the same argument. So, Subclaim 2 follows.

Since \( a \) and \( b \) are symmetric and \( |P_x| \geq 2 \), we may assume that there exists a vertex \( x_{r,j-1} \). Note that, by Lemma 1, \( x_{r,j-1} \notin A \cup B \). Let \( H \) be the set of vertices in \( x_{r,1}, P_r, x_{r,j-1} \). Note that, for any \( h \in H \), \( h \notin Int(A) \) and \( h \notin Int(B) \). Let \( D_0 \) be

\[
D_0 := I(H, H) \cup \{x_{r,j}\}
\]

and, for \( z \geq 1 \), let \( D_z \) be

\[
D_z := D_{z-1} \cup I(\text{Int}(D_{z-1}), H).
\]

\( D_{z-1} \) will be interpreted as \( \emptyset \).
Suppose \( d_i \in D_i \) and \( b_j \in B_j, a_k \in A_k \), for some \( x_i, y_j \geq 0 \).

Note that \( x_{r,j} \in A \cap B \), that is, \( x_{r,j} \in A_k \) and \( x_{r,j} \in B_j \) for some \( x, y \geq 0 \).

Also, note that \( i' \neq 1, k-1 \) by Subclaim 2.

We prove the following statements.

1. **There do not exist two distinct vertices** \( d_i \) and \( b_j \) in \( P_i \) such that \( d_i \in D_i \) and \( b_j \in B_j \) for any \( z \geq 0 \) and for \( i = 1, \ldots, k-1, i \neq i' \).

2. **There do not exist two distinct vertices** \( a_i \) and \( d_i \) in \( P_{k-1} \), such that \( a_i \in A_i \) and \( d_i \in D_i \), for any \( z \geq 0 \).

**Proof.** If there exist such vertices \( d_i \) and \( b_j \) in \( P_i \), then choosing \( z \) minimal, and considering three paths as follows: If \( i < i' \), then we can get

   - (a) \( a, P, b, d, P, x_{r,j-1} \) and \( a, P, b \).

   - (b) \( a, P, d, b, P, x_{r,j-1} \) and \( a, P, b \).

If \( i > i' \), then we can get

   - (c) \( a, P, x_{r,j-1} \) and \( a, P, d, b, P, b \).

   - (d) \( a, P, x_{r,j-1} \) and \( a, P, b, d, P, b \).

If there exists a vertex \( d_i \) in \( P_{k-1} \), then we choose \( z \) minimal and consider three paths as

   - (e) \( a, P, x_{r,j-1} \) and \( b, P, d, a, P, b \).

   - (f) \( a, P, x_{r,j-1} \) and \( b, P, a, d, P, b \).

To prove those statements, it is sufficient to prove the following subclaim.

**Subclaim 3.** If there exist three paths \( l_1, l_2, \) and \( l_3 \) in the following cases:

   **Case 1.** \( l_1 \) is connecting from \( a \) to \( x_{r,j-1} \), \( l_2 \) is connecting from \( d_i \) to \( b_j \), and \( l_3 \) is connecting from \( a \) to \( b \).

   **Case 2.** \( l_1 \) is connecting from \( a \) to \( x_{r,j-1} \), \( l_2 \) is connecting from \( d_i \) to \( b_j \), and \( l_3 \) is connecting from \( a \) to \( b \).

   **Case 3.** \( l_1 \) is connecting from \( a \) to \( x_{r,j-1} \), \( l_2 \) is connecting from \( d_i \) to \( a_k \), and \( l_3 \) is connecting from \( b_j \) to \( b \).

   **Case 4.** \( l_1 \) is connecting from \( a \) to \( d_i \), \( l_2 \) is connecting from \( a_k \) to \( x_{r,j-1} \), and \( l_3 \) is connecting from \( b_j \) to \( b \).

   **Case 5.** \( l_1 \) is connecting from \( a \) to \( d_i \), \( l_2 \) is connecting from \( a_k \) to \( b_j \), and \( l_3 \) is connecting from \( b_j \) to \( x_{r,j-1} \).

   **Case 6.** \( l_1 \) is connecting from \( a \) to \( b_j \), \( l_2 \) is connecting from \( d_i \) to \( x_{r,j-1} \), and \( l_3 \) is connecting from \( a_k \) to \( b \).
Case 7. \( l_1 \) is connecting from \( a \) to \( b_y \), \( l_2 \) is connecting from \( d_x \) to \( a_x \), and \( l_3 \) is connecting from \( b \) to \( x_{r,j-1} \).

Case 8. \( l_1 \) is connecting from \( a \) to \( b_y \), \( l_2 \) is connecting from \( d_x \) to \( b \), and \( l_1 \) is connecting from \( a_x \) to \( x_{r,j-1} \).

Case 9. \( l_1 \) is connecting from \( a \) to \( x_{r,j-1} \), \( l_2 \) is connecting from \( b_y \) to \( x_{r,j-1} \), and \( l_3 \) is connecting from \( b_y \) to \( b \).

Case 10. \( l_1 \) is connecting from \( a \) to \( a_x \), \( l_2 \) is connecting from \( d_x \) to \( x_{r,j-1} \), and \( l_3 \) is connecting from \( b_y \) to \( x_{r,j-1} \).

And also, \( l_1 \cup l_2 \cup l_3 \) satisfies the following conditions:

(S1) \( l_1 \cup l_2 \cup l_3 \) includes all the edges in \( L \) and all the vertices in \( \text{Int}(A_{r-1}) \), \( \text{Int}(B_{r-1}) \), \( \text{Int}(D_{r-1}) \), and \( H \).

(S2) The only vertices of \( l_1 \cup l_2 \cup l_3 \) not in \( P \) occur in segment of \( l_1 \setminus L \), \( l_2 \setminus L \), and \( l_3 \setminus L \) of the form \( w, w * r, r \), where \( w \) and \( r \) are both in \( P \) but not both in \( D_r \).

(S3) For each of the paths of \( l_1 \setminus L \), \( l_2 \setminus L \), and \( l_3 \setminus L \), say \( Q_i \), and each \( z' \leq z - 1 \), if there is a vertex \( q \) such that \( q \in Q_i \cap \text{Int}(D_r) \), then there are two vertices of \( \text{Fr}(D_r) \) occurring before and after \( q \) along \( Q_i \), and each of the vertices between them along \( Q_i \) is in \( \text{Int}(D_r) \) and if \( Q_i \) contains \( x_{r,j-1} \), then \( x_{r,j-1} \) is adjacent to \( H \) in \( Q_i \) and \( Q_i \) contains the segment \( x_{r,1}, P, x_{r,j-1} \).

Then there exist one or two disjoint circuits that contain all the edges in \( L \).

Proof. We prove Subclaim 3 by induction on \( z \). Suppose \( z = 0 \).

Let \( T_x \) be a path \( T_x = q * d_0, q \in H \). \( T_x \) does not intersect any segment \( w \), \( w * r, r \) in \( l_1 \) or in \( l_2 \) or in \( l_3 \) with \( w \) and \( r \) in \( P \) except for its end vertices \( d_0 \) and \( q \). For otherwise, both \( w \) and \( r \) are in \( D_r \), which is contrary to (S2). By the condition (S3), there exists the vertex \( x_{r,j-1} \) which is preceding \( q \) such that the segment \( q, l_1, x_{r,j-1} \) or \( x_{r,j-1}, l_1, q \) or \( q, l_2, x_{r,j-1} \) or \( x_{r,j-1}, l_2, q \) or \( q, l_3, x_{r,j-1} \) or \( x_{r,j-1}, l_3, q \) does not contain edges in \( L \). Also, by the condition (S3), it is easy to check that the segment \( q, l_1, x_{r,j-1} \) or \( x_{r,j-1}, l_1, q \) or \( q, l_2, x_{r,j-1} \) or \( x_{r,j-1}, l_2, q \) or \( q, l_3, x_{r,j-1} \) or \( x_{r,j-1}, l_3, q \) does not contain vertices in \( \text{Int}(A_{r-1}) \cup \text{Int}(B_{r-1}) \).

We consider eleven cases for \( l_1, l_2, \) and \( l_3 \).

Case 1. \( l_1 \) is connecting from \( a \) to \( x_{r,j-1} \), \( l_2 \) is connecting from \( d_0 \) to \( b_y \), and \( l_3 \) is connecting from \( a_x \) to \( b \).

In this case, we get two paths \( a, l_1, q, q * d_0, d_0, l_2, b_y \) and \( l_3 \) which satisfy Claim 1, and hence, the result follows.
Case 2. \( l_1 \) is connecting from \( a \) to \( x_{r,j-1} \), \( l_2 \) is connecting from \( d_0 \) to \( b \), and \( l_3 \) is connecting from \( a_s \) to \( b_j \).

In this case, we get two paths \( a, l_1, q, q \ast d_0, d_0, l_2, b \) and \( l_3 \) which satisfy Claim 1, and hence, the result follows.

Case 3. \( l_1 \) is connecting from \( a \) to \( x_{r,j-1} \), \( l_2 \) is connecting from \( d_0 \) to \( a_s \), and \( l_3 \) is connecting from \( b_j \) to \( b \).

In this case, we get two paths \( a, l_1, q, q \ast d_0, d_0, l_2, a_s \) and \( l_3 \) which satisfy Claim 1, and hence, the result follows.

Case 4. \( l_1 \) is connecting from \( a \) to \( d_0 \), \( l_2 \) is connecting from \( a_s \) to \( x_{r,j-1} \), and \( l_3 \) is connecting from \( b \) to \( b \).

In this case, we get two paths \( a, l_1, d_0, d_0 \ast q, q, \bar{T}_2, a_s \) and \( l_3 \) which satisfy Claim 1, and hence, the result follows.

Case 5. \( l_1 \) is connecting from \( a \) to \( d_0 \), \( l_2 \) is connecting from \( a_s \) to \( b_j \), and \( l_3 \) is connecting from \( b \) to \( x_{r,j-1} \).

In this case, we get two paths \( a, l_1, d_0, d_0 \ast q, q, \bar{T}_2, b \) and \( l_2 \) which satisfy Claim 1, and hence, the result follows.

Case 6. \( l_1 \) is connecting from \( a \) to \( b_j \), \( l_2 \) is connecting from \( d_0 \) to \( x_{r,j-1} \), and \( l_3 \) is connecting from \( a_s \) to \( b \).

In this case, we get a cycle \( d_0, l_2, q, q \ast d_0, d_0 \) and two paths \( l_1 \) and \( l_3 \), which satisfy Subclaim 1, and hence, the result follows.

Case 7. \( l_1 \) is connecting from \( a \) to \( b_j \), \( l_2 \) is connecting from \( d_0 \) to \( a_s \), and \( l_3 \) is connecting from \( b \) to \( x_{r,j-1} \).

In this case, we get two paths \( l_1 \) and \( a_s, \bar{T}_2, d_0, d_0 \ast q, q, \bar{T}_2, b \) which satisfy Claim 1, and hence, the result follows.

Case 8. \( l_1 \) is connecting from \( a \) to \( b_j \), \( l_2 \) is connecting from \( d_0 \) to \( b \), and \( l_3 \) is connecting from \( a_s \) to \( x_{r,j-1} \).

In this case, we get two paths \( l_1 \) and \( b, \bar{T}_2, d_0, d_0 \ast q, q, \bar{T}_2, a \) which satisfy Claim 1, and hence, the result follows.

Case 9. \( l_1 \) is connecting from \( a \) to \( d_0 \), \( l_2 \) is connecting from \( b_j \) to \( x_{r,j-1} \), and \( l_3 \) is connecting from \( a_s \) to \( b \).

In this case, we get two paths \( l_1 \) and \( a, l_1, d_0, d_0 \ast q, q, \bar{T}_2, b \) which satisfy Claim 1, and hence, the result follows.

Case 10. \( l_1 \) is connecting from \( a \) to \( a_s \), \( l_2 \) is connecting from \( d_0 \) to \( x_{r,j-1} \), and \( l_3 \) is connecting from \( b_j \) to \( b \).

In this case, we get two paths \( l_1 \) and \( \bar{T}_2, d_0, d_0 \ast q, q, \bar{T}_2, b \), and a cycle \( d_0, l_2, q, q \ast d_0 \) which satisfy Subclaim 1, and hence, the result follows.

Case 11. \( l_1 \) is connecting from \( a \) to \( a_s \), \( l_2 \) is connecting from \( d_0 \) to \( b \), and \( l_3 \) is connecting from \( b_j \) to \( x_{r,j-1} \).
In this case, we get two paths \( l_1 \) and \( b, \bar{T}_2, d_{o_1}, d_{q_1} \), which satisfy Claim 1, and hence, the result follows.

Suppose \( z > 0 \). If \( d_{z} \in D_{z-1} \), the result follows by the induction hypothesis. So, we may assume \( d_{z} \in D_{z} \setminus D_{z-1} \). We can choose a path \( d_{z} \ast y_{z-1} \), connecting \( d_{z} \) to \( y_{z-1} \), where \( y_{z-1} \in \text{Int}(D_{z-1}) \). This path does not intersect any segment \( w, w \ast r, r \) in \( l_{1} \), or in \( l_{2} \) or in \( l_{3} \) with \( w \) and \( r \) in \( P \) except for its end vertices \( d_{z} \) and \( y_{z-1} \). For otherwise, both \( w \) and \( r \) are in \( D_{z} \), which is contrary to \((S)\). By the condition \((S)\), there exists a vertex \( d_{z-1} \in D_{z-1} \) which is preceding \( y_{z-1} \) such that the segment \( d_{z-1} \ast l_{1}, y_{z-1} \) or \( d_{z-1} \ast l_{2}, y_{z-1} \) or \( d_{z-1} \ast l_{3}, y_{z-1} \) does not contain edges in \( L \). Now we choose a vertex \( d_{z} \) which is the last vertex before \( y_{z-1} \) along \( l_{1} \) in \( Fr(D_{z}) \) (if \( y_{z-1} \in l_{1} \)) or along \( l_{2} \) in \( Fr(D_{z}) \) (if \( y_{z-1} \in l_{2} \)) or along \( l_{3} \) in \( Fr(D_{z}) \) (if \( y_{z-1} \in l_{3} \)) and \( d_{z} \) is in \( Fr(D_{z}) \) for any \( z' \leq z - 1 \), and choose \( z' \) minimal so that \( d_{z} \notin \text{Cl}(D_{z-1}) \). Also, by the condition \((S)\), there exists \( d_{z-1} \in D_{z-1} \) which is succeeding \( y_{z-1} \) such that the segment \( y_{z-1} \ast l_{1}, d_{z-1} \ast l_{1}, l_{3}, y_{z-1} \) or \( y_{z-1} \ast l_{2}, d_{z-1} \ast l_{2}, l_{3}, y_{z-1} \) or \( y_{z-1} \ast l_{3}, d_{z-1} \ast l_{3}, l_{3}, y_{z-1} \) does not contain edges in \( L \). Now we choose a vertex \( d_{z} \) which is the last vertex after \( y_{z-1} \) along \( l_{1} \) in \( Fr(D_{z}) \) (if \( y_{z-1} \in l_{1} \)) or along \( l_{2} \) in \( Fr(D_{z}) \) (if \( y_{z-1} \in l_{2} \)) or along \( l_{3} \) in \( Fr(D_{z}) \) (if \( y_{z-1} \in l_{3} \)) and \( d_{z} \) is in \( Fr(D_{z}) \) for any \( z' \leq z - 1 \), and choose \( z' \) minimal so that \( d_{z} \notin \text{Cl}(D_{z-1}) \). We will write \( d_{z} \) instead of \( d_{z} \ast y_{z-1} \) since it may not be confusing for readers.

Then there does not exist a vertex that is in \( \text{Int}(D_{z-1}) \) in the segments both \( y_{z-1} \ast l_{1}, d_{z} \ast l_{1}, y_{z-1} \) (if \( y_{z-1} \in l_{1} \)), or both \( y_{z-1} \ast l_{1}, d_{z} \ast l_{1}, y_{z-1} \ast l_{2}, d_{z} \ast l_{2}, y_{z-1} \ast l_{3}, d_{z} \ast l_{3}, y_{z-1} \) (if \( y_{z-1} \in l_{1} \)).

We may assume that there are no vertices in \( \text{Int}(A) \cup \text{Int}(B) \) in the segments both \( y_{z-1} \ast l_{1}, d_{z} \ast l_{1}, y_{z-1} \ast l_{2}, d_{z} \ast l_{2}, y_{z-1} \ast l_{3}, d_{z} \ast l_{3}, y_{z-1} \) (if \( y_{z-1} \in l_{1} \)), or both \( y_{z-1} \ast l_{1}, d_{z} \ast l_{1}, y_{z-1} \ast l_{2}, d_{z} \ast l_{2}, y_{z-1} \ast l_{3}, d_{z} \ast l_{3}, y_{z-1} \) (if \( y_{z-1} \in l_{1} \)). For otherwise, there exist some \( P \) of \( P \) which contains distinct vertices \( r, w \) such that \( r \in D_{z} \) and \( w \in A \cup B \). But, in this case, we can take three paths which satisfy the case \( z' \). Hence the result follows by the induction hypothesis.

Now we consider eleven cases for \( l_{1}, l_{2}, \) and \( l_{3} \).

**Case 1.** \( l_{1} \) is connecting from \( a \) to \( x_{r', j-1} \), \( l_{2} \) is connecting from \( d_{z} \) to \( b_{r} \), and \( l_{3} \) is connecting from \( a_{r} \) to \( b \).

In this case, if \( y_{z-1} \in l_{2} \), then we can replace the path \( l_{2} \) such that \( d_{z}, \bar{T}_{2}, d_{z}, d_{z} \ast y_{z-1}, l_{2}, b_{r}, l_{3} \) and \( l_{3} \) are still \( l_{1} \) and \( l_{3} \). These three paths \( l_{1}, l_{2}, \) and \( l_{3} \) satisfy the case \( z' \) of Case 1. So, the result follows by the induction hypothesis.

If \( y_{z-1} \in l_{1} \), then we can replace the path \( l_{1} \) such that \( a_{r}, l_{1}, y_{z-1}, y_{z-1} \ast d_{z}, l_{2}, b_{r}, l_{3} \), and also we can replace the path \( l_{1} \) such that \( d_{z}, l_{1}, x_{r', j-1}, l_{3} \) is \( l_{1} \). These three paths \( l_{1}, l_{2}, \) and \( l_{3} \) satisfy the case \( z' \) of Case 6. So, the result follows by the induction hypothesis.
If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $d_z, T_z, a_z$ and also we can replace the path $l_1$ such that $b_z, T_z, d_z, d_z \ast y_{z-1}, y_{z-1}, l_1, b, l_1$ is still $l_1$. These three paths $l_1, l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

Case 2. $l_1$ is connecting from $a$ to $x_r, j_{-1}$, $l_2$ is connecting from $d_z$ to $b$, and $l_3$ is connecting from $a_z$ to $b_z$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $d_z, T_z, d_z, d_z \ast y_{z-1}, y_{z-1}, l_2, b, l_1$ and $l_3$ are still $l_1$ and $l_3$. These three paths $l_1, l_2$, and $l_3$ satisfy the case $z'$ of Case 2. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_1$, then we can replace the path $l_1$ such that $a, l_1, d_z$ and also we can replace the path $l_1$ such that $b_z, T_z, d_z, d_z \ast y_{z-1}, y_{z-1}, l_1, x_r, j_{-1}$. $l_2$ is $l_1$. These three paths $l_1, l_2$, and $l_3$ satisfy the case $z'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $d_z, T_z, a_z$ and also we can replace the path $l_1$ such that $b_z, T_z, y_{z-1}, y_{z-1} \ast d_z, d_z, l_2, b, l_1$ is still $l_1$. These three paths $l_1, l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

Case 3. $l_1$ is connecting from $a$ to $x_r, j_{-1}$, $l_2$ is connecting from $d_z$ to $a_z$, and $l_3$ is connecting from $b_z$ to $b$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $d_z, T_z, d_z, d_z \ast y_{z-1}, y_{z-1}, l_2, a_z, l_1$ and $l_3$ are still $l_1$ and $l_3$. These three paths $l_1, l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_1$, then we can replace the path $l_1$ such that $a_z, T_z, d_z, d_z \ast y_{z-1}, y_{z-1}, l_1, x_r, j_{-1}$ and also we can replace the path $l_1$ such that $a, l_1, d_z, l_3$ is still $l_1$. These three paths $l_1, l_2$, and $l_3$ satisfy the cases $z'$ of Case 4. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $d_z, T_z, b_z$, and also we can replace the path $l_1$ such that $a_z, T_z, d_z, d_z \ast y_{z-1}, y_{z-1}, l_1, b, l_1$ is still $l_1$. These three paths $l_1, l_2$, and $l_3$ satisfy the case $z'$ of Case 1. So, the result follows by the induction hypothesis.

Case 4. $l_1$ is connecting from $a$ to $d_z$, $l_2$ is connecting from $a_z$ to $x_r, j_{-1}$, and $l_3$ is connecting from $b_z$ to $b$.

In this case, if $y_{z-1} \in l_1$, then we can replace the path $l_1$ such that $a, l_1, y_{z-1}, y_{z-1} \ast d_z, d_z, T_z, d_z, l_2$ and $l_1$ are still $l_1$ and $l_3$. These three paths $l_1, l_2$, and $l_3$ satisfy the case $z'$ of Case 4. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_2$, then we can replace the path $l_1$ such that $a, l_1, d_z, d_z \ast y_{z-1}, y_{z-1}, l_2, x_r, j_{-1}$ and also we can replace the path $l_2$ such that $d_z, T_z, a_z, l_3$ is
still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $d_z$, $l_3$, $b$, and also we can replace the path $l_1$ such that $a$, $l_1$, $d_z$, $d_z \ast y_{z-1}$, $T_b$, $b$, $l_3$ is $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 8. So, the result follows by the induction hypothesis.

Case 5. $l_1$ is connecting from $a$ to $d_z$, $l_2$ is connecting from $a_\ast$ to $b_y$, and $l_3$ is connecting from $b$ to $x_{r,j-1}$.

In this case, if $y_{z-1} \in l_1$, then we can replace the path $l_1$ such that $a$, $l_1$, $y_{z-1}$, $y_{z-1} \ast d_z$, $d_z \ast y_{z-1}$, $l_2$ and $l_3$ are still $l_2$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_2$, then we can replace the path $l_1$ such that $a$, $l_1$, $d_z$, $d_z \ast y_{z-1}$, $l_2$, $b$, and also we can replace the path $l_3$ such that $d_z$, $T_b$, $a_\ast$, $l_3$ is still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 7. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $d_z$, $T_b$, $b$ and also we can replace the path $l_1$ such that $a$, $l_1$, $d_z$, $d_z \ast y_{z-1}$, $l_1$, $x_{r,j-1}$, $l_3$ is $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 2. So, the result follows by the induction hypothesis.

Case 6. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $d_z$ to $x_{r,j-1}$, and $l_3$ is connecting from $a_\ast$ to $b$.

In this case, if $y_{z-1} \in l_1$, then we can replace the path $l_2$ such that $d_z$, $T_b$, $d_z$, $d_z \ast y_{z-1}$, $l_2$, $x_{r,j-1}$, $l_1$ and $l_3$ are still $l_2$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 6. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_2$, then we can replace the path $l_1$ such that $a$, $l_1$, $y_{z-1}$, $y_{z-1} \ast d_z$, $d_z$, $x_{r,j-1}$, $l_1$ is still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 1. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $d_z$, $T_b$, $a_\ast$ and also we can replace the path $l_3$ such that $b$, $T_b$, $y_{z-1}$, $y_{z-1} \ast d_z$, $d_z$, $x_{r,j-1}$, $l_3$ is still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 7. So, the result follows by the induction hypothesis.

Case 7. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $d_z$ to $a_\ast$, and $l_3$ is connecting from $b$ to $x_{r,j-1}$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $d_z$, $T_b$, $d_z$, $d_z \ast y_{z-1}$, $l_2$, $a_\ast$, $l_1$ and $l_3$ are still $l_2$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 7. So, the result follows by the induction hypothesis.
If \( y_{z-1} \in l_1 \), then we can replace the path \( l_1 \) such that \( a, l_1, d_e \) and also we can replace the path \( l_1 \) such that \( a_x, T_2, d_e, d_e * y_{z-1}, y_{z-1}, l_1, b_y, l_3 \) is still \( l_3 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 5. So, the result follows by the induction hypothesis.

If \( y_{z-1} \in l_1 \), then we can replace the path \( l_1 \) such that \( d_x, T_3, b \) and also we can replace the path \( l_1 \) such that \( a_x, T_2, d_e, d_e * y_{z-1}, y_{z-1}, l_3, x_r, j-1, l_3 \) is still \( l_3 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 8. So, the result follows by the induction hypothesis.

**Case 8.** \( l_1 \) is connecting from \( a \) to \( b \), \( l_2 \) is connecting from \( d_e \) to \( b \), and \( l_3 \) is connecting from \( a_x \) to \( x_{r, j-1} \).

In this case, if \( y_{z-1} \in l_1 \), then we can replace the path \( l_2 \) such that \( d_e, T_2, d_e, d_e * y_{z-1}, y_{z-1}, l_2, b, l_1, \) and \( l_1 \) are still \( l_1, l_2, \) and \( l_3 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 8. So, the result follows by the induction hypothesis.

If \( y_{z-1} \in l_1 \), then we can replace the path \( l_1 \) such that \( b, T_1, y_{z-1}, y_{z-1} * d_e, d_e, l_2, b, l_1 \) and also we can replace the path \( l_1 \) such that \( a, l_1, d_e, l_2 \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 4. So, the result follows by the induction hypothesis.

If \( y_{z-1} \in l_1 \), then we can replace the path \( l_1 \) such that \( d_e, l_1, x_{r, j-1} \) and also we can replace the path \( l_1 \) such that \( a_x, y_{z-1}, y_{z-1} * d_e, d_e, l_2, b, l_1 \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 6. So, the result follows by the induction hypothesis.

**Case 9.** \( l_1 \) is connecting from \( a \) to \( d_e, l_2 \) is connecting from \( b \) to \( x_{r, j-1} \), and \( l_3 \) is connecting from \( a_x \) to \( b \).

In this case, if \( y_{z-1} \in l_1 \), then we can replace the path \( l_1 \) such that \( a, l_1, y_{z-1}, y_{z-1} * d_e, d_e, l_2, l_1 \) and \( l_1 \) are still \( l_1, l_2, \) and \( l_3 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 9. So, the result follows by the induction hypothesis.

If \( y_{z-1} \in l_2 \), then we can replace the path \( l_1 \) such that \( a, l_1, d_e, d_e * y_{z-1}, y_{z-1}, l_2, x_{r, j-1} \) and also we can replace the path \( l_2 \) such that \( d_e, T_2, b, l_1 \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 1. So, the result follows by the induction hypothesis.

If \( y_{z-1} \in l_3 \), then we can replace the path \( l_1 \) such that \( d_e, l_3, b \) and also we can replace the path \( l_1 \) such that \( a, l_1, d_e, d_e * y_{z-1}, y_{z-1}, T_1, a_x, l_3 \) is \( l_2 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 11. So, the result follows by the induction hypothesis.

**Case 10.** \( l_1 \) is connecting from \( a \) to \( a_x, l_2 \) is connecting from \( d_e \) to \( x_{r, j-1} \), and \( l_3 \) is connecting from \( b \) to \( b \).

In this case, if \( y_{z-1} \in l_2 \), then we can replace the path \( l_2 \) such that \( d_e, T_2, d_e, d_e * y_{z-1}, y_{z-1}, l_2, x_{r, j-1} \) and \( l_1 \) are still \( l_1, l_2, \) and \( l_3 \). These three paths
1, l_2, and l_3 satisfy the case z' of Case 10. So, the result follows by the induction hypothesis.

If y_{z-1} \in l_1, then we can replace the path l_2 such that a, l_1, y_{z-1}, y_{z-1} \ast d_z, d_z, l_2, x_{r, j-1} and also we can replace the path l_3 such that d_z, l_3, a, l_3 is still l_3. These three paths l_1, l_2, and l_3 satisfy the case z' of Case 3. So, the result follows by the induction hypothesis.

If y_{z-1} \in l_1, then we can replace the path l_2 such that d_z, l_3, b and also we can replace the path l_3 such that b, l_3, y_{z-1}, y_{z-1} \ast d_z, d_z, l_2, x_{r, j-1}, l_1 is still l_1. These three paths l_1, l_2, and l_3 satisfy the case z' of Case 11. So, the result follows by the induction hypothesis.

Case 11. l_1 is connecting from a to a, l_2 is connecting from d_z to b, and l_3 is connecting from b to x_{r, j-1}.

In this case, if y_{z-1} \in l_1, then we can replace the path l_2 such that d_z, l_2, d_z \ast y_{z-1}, y_{z-1}, l_2, b, l_1 and l_3 are still l_1 and l_3. These three paths l_1, l_2, and l_3 satisfy the case z' of Case 11. So, the result follows by the induction hypothesis.

If y_{z-1} \in l_1, then we can replace the path l_2 such that a, l_1, y_{z-1}, y_{z-1} \ast d_z, d_z, l_2, b, l_1 and l_3 are still l_1 and l_3. These three paths l_1, l_2, and l_3 satisfy the case z' of Case 9. So, the result follows by the induction hypothesis.

If y_{z-1} \in l_1, then we can replace the path l_2 such that y_{z-1}, l_2, x_{r, j-1}, y_{z-1} \ast d_z, d_z, l_2, b, l_1 is still l_1. These three paths l_1, l_2, and l_3 satisfy the case z' of Case 10. So, the result follows by the induction hypothesis.

So, Subclaim 3 follows.

Since V(P) is finite, the sequence of sets D_0 \subseteq D_1 \subseteq \cdots must be constant from some point onwards. Let D be the final sets.

By Subclaim 3, we can get the fact that there does not exist two distinct vertices m and n such that m \in D and n \in B in P, for i = 1, \ldots, k-1, i \neq i'.

Since |Fr(B)| = k-1, |Fr(D)| \leq k. Also, by Subclaim 3, we can get the fact that there does not exist two distinct vertices m' and n' in P_{k-1} such that m' \in D and n' \in A. Therefore, we can get the fact that |Fr(D)| \leq k-2. In this case, Fr(D) \cup \{x_{r-1, m-1}\} is a cutset separating H from a, or H from b, and its cardinality is at most k-1, which is contrary to the connectivity of G. So, Claim 2 follows.

We must consider two cases for k.

Case 1. k is even.

The number of segments P_i is k-1. In this case, since k-1 is odd, and by Claim 2, either |Fr(A)| \leq k-2 or |Fr(B)| \leq k-2. But either Fr(A) \cup \{x_i\} or Fr(B) \cup \{x_{i-1, m_i}\} is a cutset separating a from b. Thus G has a cutset of cardinality at most k-1, which is contrary to the connectivity of G.
Case 2. $k$ is odd.

Since $k - 1$ is even, we only consider the case that both $|Fr(A)|$ and $|Fr(B)|$ are $k - 1$. That is, there exist at least two vertices in $A$ or in $B$ for all $P_i$.

Since $k$ is odd and $G - L$ is connected, there exists at least one path which is connecting from some segment $P_i$ which have two vertices of $A$, to some other segment $P_j$ which have two vertices of $B$. Let $l = x_{r,i} * x_{r,k}$.

Note that $x_{r,i}$ is not in Int($A$) and $x_{r,k}$ is not in Int($B$). $P(l)$ denotes a path along $P$ from $x_{r,i}$ to $x_{r,k}$, and also, $P'(l)$ denotes a path $P(l) \setminus \{x_{r,i}, x_{r,k}\}$.

We prove the following facts.

**Fact 1.** $i' < i''$.

**Proof.** Assume not. We consider three cases.

**Case 1.** The path $x_{r,i+1}, P_r, x_{r,m}$ contains a vertex $a_c \in A_c$ and the path $x_{r,i+1}, P_r, x_{r,k-1}$ contains a vertex $b_y \in B_y$.

In this case, we can take two paths $a, b, b_y$ and $a_c$, $P, b$, and a cycle $x_{r,k}, P, x_{r,j_1}, x_{r,j} * x_{r,k}, x_{r,k}$, which satisfy Subclaim 1. So, the result follows.

**Case 2.** The path $x_{r,1}, P_r, x_{r,j-1}$ contains a vertex $a_c \in A_c$ and the path $x_{r,1}, P_r, x_{r,k-1}$ contains a vertex $b_y \in B_y$.

Since $a$ and $b$ are symmetric, we consider only the case that the path $x_{r,1}, P_r, x_{r,j-1}$ contains a vertex $a_c \in A_c$ and the path $x_{r,1}, P_r, x_{r,k-1}$ contains a vertex $b_y \in B_y$.

Then we can take two paths $a, b_y$ and $a_c, P, x_{r,k}, x_{r,k} * x_{r,j}, x_{r,j}, P, b$, which satisfy Claim 1. So, the result follows.

**Case 3.** The path $x_{r,1}, P_r, x_{r,j-1}$ contains a vertex $a_c \in A_c$ and the path $x_{r,k+1}, P_r, x_{r,m}$ contains a vertex $b_y \in B_y$.

In this case, we can take two paths $a, b, x_{r,k}, x_{r,k} * x_{r,j_1}, x_{r,j}, P, b$ and $b, P, a_c$, which satisfy Claim 1. So, the result follows.

**Fact 2.** There does not exist a vertex $a_c \in A_c$ in $x_{r,j+1}, P_r, x_{r,m}$, and there does not exist a vertex $b_y \in B_y$ in $x_{r,1}, P_r, x_{r,k-1}$.

**Proof.** Assume not. Since $a$ and $b$ are symmetric, so we consider only the case that there exists a vertex $a_c \in A_c$ in $x_{r,j+1}, P_r, x_{r,m}$. We consider two cases.

**Case 1.** There exists a vertex $b_y \in B_y$ in $x_{r,1}, P_r, x_{r,k-1}$.

In this case, we can take two paths $a, b, x_{r,j_1}, x_{r,j} * x_{r,k}, x_{r,k}, P, b$ and $a_c, P, b_y$, which satisfy Claim 1. So, the result follows.
Case 2. There exists a vertex \( b_j \in B_j \) in \( x_{r',k+1}, P_{r'}, x_{r,m'} \).

In this case, we can take two paths \( a, P, x_{r,j'}, x_{r,j} \ast x_{r,k}, x_{r',k}, \bar{P}, a \) and \( b_j, P, b_j \), which satisfy Claim 1. So, the result follows.

We choose a path \( l \) and \( P \) so that \( N(x_{r,j}, x_{r,k}) \) is as large as possible. We prove the following claim.

Claim 3. \( N(x_{r,j}, x_{r,k}) \geq 2 \).

Proof. Assume to the contrary, that is, \( N(x_{r,j}, x_{r,k}) = 1 \). We may assume that there exists a vertex \( u \) in \( x_{r,j+1}, P_{r'}, x_{r,m'} \) or there exists a vertex \( v \) in \( x_{r,1}, P_{r'}, x_{r,k-1} \). For otherwise, if \( j = m_r \) and \( k = 1 \), then \( |l| \geq 3 \).

So, there must exist a vertex \( w \) in \( \{x_{r,j}, x_{r,k}\} \). But, \( \{x_{r,j}, x_{r,k}\} \) is a cutset separating \( w \) from \( a \), or \( w \) from \( b \), and since \( k \geq 4 \), the result easily follows.

Since \( a \) and \( b \) are symmetric, we may assume that there exists a vertex \( u \) in \( x_{r,j+1}, P_{r'}, x_{r,m'} \). Let \( U \) be the set of vertices in \( x_{r,j+1}, P_{r'}, x_{r,m'} \).

We re-choose a path \( l \) and \( P \) so that \( N(x_{r,j}, x_{r,k}) \) is as large as possible, and subject to that condition, \( |x_{r,1}, P_{r'}, x_{r,k}| \) is as small as possible, but \( U \neq \emptyset \). Note that the proofs before Claim 3 do not depend on the choice of \( P \).

Let \( H_0 \) be

\[
H_0 := I(U, U) \cup \{x_{r,j}\}
\]

and for \( z \geq 1 \), let \( H_z \) be

\[
H_z := H_{z-1} \cup I(\text{Int}(H_{z-1}), U).
\]

\( H_{-1} \) will be interpreted as \( \emptyset \).

Suppose \( h_i \in H_y \) and \( b_j \in B_y \), \( a_k \in A_y \) for some \( x, y \geq 0 \).

We prove the following subclaim.

Subclaim 4. If there exist three paths \( l_1, l_2, \) and \( l_3 \) in the following cases:

Case 1. \( l_1 \) is connecting from \( a \) to \( x_{r,j+1}, l_2 \) is connecting from \( h_i \) to \( b_j \), and \( l_3 \) is connecting from \( a_k \) to \( b \).

Case 2. \( l_1 \) is connecting from \( a \) to \( x_{r,j+1}, l_2 \) is connecting from \( h_i \) to \( b_j \), and \( l_3 \) is connecting from \( a_k \) to \( b_j \).

Case 3. \( l_1 \) is connecting from \( a \) to \( x_{r,j+1}, l_2 \) is connecting from \( h_i \) to \( a_k \), and \( l_3 \) is connecting from \( b_j \) to \( b \).

Case 4. \( l_1 \) is connecting from \( a \) to \( h_i, l_2 \) is connecting from \( a_k \) to \( x_{r,j+1}, \) and \( l_3 \) is connecting from \( b_j \) to \( b \).
Case 5. $l_1$ is connecting from $a$ to $h_x$, $l_2$ is connecting from $a_x$ to $b_y$, and $l_3$ is connecting from $b$ to $x_{r, j+1}$.

Case 6. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $h_x$ to $x_{r, j+1}$, and $l_3$ is connecting from $a_x$ to $b$.

Case 7. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $h_x$ to $a_x$, and $l_3$ is connecting from $b$ to $x_{r, j+1}$.

Case 8. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $h_x$ to $b$, and $l_3$ is connecting from $a_x$ to $x_{r, j+1}$.

Case 9. $l_1$ is connecting from $a$ to $a_x$, $l_2$ is connecting from $h_x$ to $b_y$, and $l_3$ is connecting from $b$ to $x_{r, j+1}$.

Case 10. $l_1$ is connecting from $a$ to $a_x$, $l_2$ is connecting from $h_x$ to $b_y$, and $l_3$ is connecting from $b$ to $x_{r, j+1}$.

Case 11. $l_1$ is connecting from $a$ to $h_x$, $l_2$ is connecting from $b_y$ to $x_{r, j+1}$, and $l_3$ is connecting from $a_x$ to $b$.

And also, $l_1 \cup l_2 \cup l_3$ satisfies the following conditions:

(S1) $l_1 \cup l_2 \cup l_3$ includes all the edges in $L$ and all the vertices in $\text{Int}(A_{z-1})$, in $\text{Int}(B_{p-1})$, in $\text{Int}(H_{z-1})$ and in $U$.

(S2) The only vertices of $l_1 \cup l_2 \cup l_3$, not in $P$ occur in segment of $l_1 \setminus L$, $l_2 \setminus L$, and $l_3 \setminus L$ of the form $w, w \ast r, r$, where $w$ and $r$ are both in $P$ but not both in $H_z$.

(S3) For each of the paths of $l_1 \setminus L, l_2 \setminus L$ and $l_3 \setminus L$, say $Q_i$, and each $z \leq z-1$, if there is a vertex $q$ such that $q \in Q_i \cap \text{Int}(H_z)$, then there are two vertices of $\text{Fr}(H_z)$ occurring before and after $q$ along $Q_i$, and each of the vertices between them along $Q_i$ is in $\text{Int}(H_z)$ and if $Q_i$ contains $x_{r, j}$, then, $x_{r, j}$ is adjacent to $U$ in $Q_i$ and $Q_i$ contains the segment $x_{r, j+1}$, $P_{r, x_{r, m}}$.

Then there exist one or two disjoint circuits that contain all the edges in $L$.

Proof. We prove Subclaim 4 by induction on $z$. Suppose $z = 0$.

Let $T_z$ be a path from $U$ to $h_0$. Also, let $q$ be the vertex such that $T_z = q \ast h_0$. $T_z$ does not intersect any segment $w, w \ast r, r$ in $l_1$, or in $l_2$ or in $l_3$ with $w$ and $r$ in $P$ except for its end vertices $h_0$ and $q$. For otherwise, both $w$ and $r$ are in $H_0$, contrary to (S2). By the condition (S2), there exists the vertex $x_{r, j+1}$, which is preceding or succeeding $q$ such that the segment $q, l_1, x_{r, j+1}$ or $x_{r, j+1}, l_1, q$ or $q, l_2, x_{r, j+1}$ or $x_{r, j+1}, l_2, q$ or $q, l_3, x_{r, j+1}$ or $x_{r, j+1}, l_3, q$ does not contain edges in $L$. Also, by the condition (S3), it is easy to check that the segment $q, l_1, x_{r, j+1}$ or $x_{r, j+1}, l_1, q$ or $q, l_2, x_{r, j+1}$ or $x_{r, j+1}, l_2, q$ or $q, l_3, x_{r, j+1}$ or $x_{r, j+1}, l_3, q$ does not contain vertices in $\text{Int}(A_{z-1}) \cup \text{Int}(B_{p-1})$.

We consider eleven cases for $l_1, l_2$, and $l_3$.

Case 1. $l_1$ is connecting from $a$ to $x_{r, j+1}$, $l_2$ is connecting from $h_0$ to $b_y$, and $l_3$ is connecting from $a_x$ to $b$. 
In this case, we get two paths \( a, l_1, q, q \ast h_0, h_0, l_2, b_x \) and \( l_3 \) which satisfy Claim 1, and hence, the result follows.

**Case 2.** \( l_1 \) is connecting from \( a \) to \( x_r, j_x + 1 \), \( l_2 \) is connecting from \( h_0 \) to \( b_x \), and \( l_3 \) is connecting from \( a_s \) to \( b_y \).

In this case, we get two paths \( a, l_1, q, q \ast h_0, h_0, l_2, b_x \) and \( l_3 \) which satisfy Claim 1, and hence, the result follows.

**Case 3.** \( l_1 \) is connecting from \( a \) to \( x_r, j_x + 1 \), \( l_2 \) is connecting from \( h_0 \) to \( a_s \), and \( l_3 \) is connecting from \( b_y \) to \( b \).

In this case, we get two paths \( a, l_1, q, q \ast h_0, h_0, l_2, a_s \) and \( l_3 \) which satisfy Claim 1, and hence, the result follows.

**Case 4.** \( l_1 \) is connecting from \( a \) to \( h_0 \), \( l_2 \) is connecting from \( a_s \) to \( x_r, j_x + 1 \), and \( l_3 \) is connecting from \( b_y \) to \( b \).

In this case, we get two paths \( a, l_1, h_0, h_0 \ast q, q, \bar{r}_x, a_s \) and \( l_3 \) which satisfy Claim 1, and hence, the result follows.

**Case 5.** \( l_1 \) is connecting from \( a \) to \( h_0 \), \( l_2 \) is connecting from \( a_s \) to \( b_y \), and \( l_3 \) is connecting from \( b_y \) to \( x_r, j_x + 1 \).

In this case, we get two paths \( a, l_1, h_0, h_0 \ast q, q, \bar{r}_x, b \) and \( l_2 \) which satisfy Claim 1, and hence, the result follows.

**Case 6.** \( l_1 \) is connecting from \( a \) to \( b_y \), \( l_2 \) is connecting from \( h_0 \) to \( x_r, j_x + 1 \), and \( l_3 \) is connecting from \( a_s \) to \( b \).

In this case, we get a cycle \( h_0, l_2, q, q \ast h_0, h_0 \) and two paths \( l_1 \) and \( l_3 \), which satisfy Subclaim 1, and hence, the result follows.

**Case 7.** \( l_1 \) is connecting from \( a \) to \( b_y \), \( l_2 \) is connecting from \( h_0 \) to \( a_s \), and \( l_3 \) is connecting from \( b \) to \( x_r, j_x + 1 \).

In this case, we get two paths \( l_1 \) and \( a_s, \bar{r}_x, h_0, h_0 \ast q, q, \bar{r}_x, b \) which satisfy Claim 1, and hence, the result follows.

**Case 8.** \( l_1 \) is connecting from \( a \) to \( b_y \), \( l_2 \) is connecting from \( h_0 \) to \( b \), and \( l_3 \) is connecting from \( a_s \) to \( x_r, j_x + 1 \).

In this case, we get two paths \( l_1 \) and \( b, \bar{r}_x, h_0, h_0 \ast q, q, \bar{r}_x, a \) which satisfy Claim 1, and hence, the result follows.

**Case 9.** \( l_1 \) is connecting from \( a \) to \( a_s \), \( l_2 \) is connecting from \( h_0 \) to \( b \), and \( l_3 \) is connecting from \( b_y \) to \( x_r, j_x + 1 \).

In this case, we get two paths \( l_1 \) and \( b, \bar{r}_x, h_0, h_0 \ast q, q, \bar{r}_x, b \) which satisfy Claim 1, and hence, the result follows.

**Case 10.** \( l_1 \) is connecting from \( a \) to \( a_s \), \( l_2 \) is connecting from \( h_0 \) to \( b_y \), and \( l_3 \) is connecting from \( b \) to \( x_r, j_x + 1 \).
In this case, we get two paths \( l_1 \) and \( b_y, T_z, h_0, h_0 \cdot q, q, T_z, b \) which satisfy Claim 1, and hence, the result follows.

**Case 1.** \( l_1 \) is connecting from \( a \) to \( h_0 \), \( l_2 \) is connecting from \( b_y \) to \( x_{i_{j+1}} \), and \( l_3 \) is connecting from \( a \) to \( b \).

In this case, we get two paths \( l_1 \) and \( a, l_1, h_0, h_0 \cdot q, q, T_z, b_y \) which satisfy Claim 1, and hence, the result follows.

Suppose \( z > 0 \). If \( h \in H_{z-1} \), the result follows by induction hypothesis. So, we may assume \( h \in H_1 \setminus H_{z-1} \). We can choose a path \( h \cdot y_{z-1} \) connecting \( h \) to \( y_{z-1} \), where \( y_{z-1} \in \text{Int}(H_{z-1}) \). This path does not intersect any segment \( w, w \cdot r, r \cdot l_1 \) or in \( l_2 \) or in \( l_3 \) with \( w \) and \( r \) in \( P \) except for its end vertices \( h \) and \( y_{z-1} \). For otherwise, both \( w \) and \( r \) are in \( H_z \), contrary to \((S_f)\). By the condition \((S_f)\), there exists a vertex \( h_{z-1} \in H_{z-1} \) which is preceding \( y_{z-1} \) such that the segment \( h_{z-1}, l_1, y_{z-1} \) or \( h_{z-1}, l_2, y_{z-1} \) or \( h_{z-1}, l_3, y_{z-1} \) does not contain edges in \( L \). Now we choose a vertex \( h_f \) which is the last vertex before \( y_{z-1} \) along \( l_1 \) (if \( y_{z-1} \in l_1 \)) or along \( l_2 \) (if \( y_{z-1} \in l_2 \)) or along \( l_3 \) (if \( y_{z-1} \in l_3 \)) and \( h_f \) is in \( \text{Fr}(H_f) \) for any \( z' < z-1 \), and choose \( z' \) minimal so that \( h_f \notin \text{Cl}(H_{z-1}) \). Also, by the condition \((S_f)\), there exists a vertex \( h'_{z-1} \in H_{z-1} \) which is succeed- ing \( y_{z-1} \) such that the segment \( h_{z-1}, l_1, h'_{z-1}, l_2, h', l_3, y_{z-1} \) does not contain edges in \( L \). Now we choose a vertex \( h_f' \) which is the last vertex after \( y_{z-1} \) along \( l_1 \) (if \( y_{z-1} \in l_1 \)) or along \( l_2 \) (if \( y_{z-1} \in l_2 \)) or along \( l_3 \) (if \( y_{z-1} \in l_3 \)) and \( h_f' \) is in \( \text{Fr}(H_f) \) for any \( z' < z-1 \), and choose \( z' \) minimal so that \( h_f' \notin \text{Cl}(H_{z-1}) \). We will write \( h_f' \) instead of \( h_f' \) since it may not be confusing for readers.

Then there does not exist a vertex that is in \( \text{Int}(H_{z-1}) \) in the segments both \( y_{z-1}, h, h', l_1, y_{z-1} \) (if \( y_{z-1} \in l_1 \)), or both \( y_{z-1}, h, h', l_2, y_{z-1} \) (if \( y_{z-1} \in l_2 \)), or both \( y_{z-1}, h, h', l_3, y_{z-1} \) (if \( y_{z-1} \in l_3 \)).

We may assume that there are no vertices in \( \text{Int}(A) \cup \text{Int}(B) \) in the segments both \( y_{z-1}, h, h$, and \( h, l_1, y_{z-1} \) (if \( y_{z-1} \in l_1 \)), or both \( y_{z-1}, l_2, h, h$, \( y_{z-1} \)) (if \( y_{z-1} \in l_2 \)), or both \( y_{z-1}, l_3, h, h$, \( y_{z-1} \)) (if \( y_{z-1} \in l_3 \)). For otherwise, there exist some \( P_i \) of \( P \) which contains distinct vertices \( r, w \) such that \( r \in H_z \) and \( w \in A \cup B \). But, in this case, we can take three paths which satisfy the case \( z' \). Hence the result follows by the induction hypothesis.

Now we consider eleven cases for \( l_1, l_2, \) and \( l_3 \).

**Case 1.** \( l_1 \) is connecting from \( a \) to \( x_{i_{j+1}} \), \( l_2 \) is connecting from \( h_0 \) to \( b_y \), and \( l_3 \) is connecting from \( a \) to \( b \).

In this case, if \( y_{z-1} \in l_1 \), then we can replace the path \( l_2 \) such that \( h_0, \) \( T_z, h_0, h_0 \cdot q, q, T_z, b \) and \( l_1 \) are still \( l_1 \) and \( l_2 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( z' \) of Case 1. So, the result follows by the induction hypothesis.

If \( y_{z-1} \in l_1 \), then we can replace the path \( l_1 \) such that \( h, l_3, y_{z-1}, y_{z-1} \cdot h, h, l_2, b_y \) and also we can replace the path \( l_1 \) such that \( h, l_3, x_{i_{j+1}}, l_3 \) is \( l_3 \).
These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 6. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $h_z, T_z, a$, and also we can replace the path $l_1$ such that $b_1, T_z, h_z, h_z \cdot y_{z-1}, y_{z-1}, l_1, b, l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

Case 2. $l_1$ is connecting from $a$ to $x_{r,j+1}$, $l_2$ is connecting from $h_z$ to $b$, and $l_3$ is connecting from $a_s$ to $b_s$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $h_z, T_z, h_z, h_z \cdot y_{z-1}, y_{z-1}, l_2, b, l_1$ and $l_1$ are still $l_1$ and $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 2. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_1$ such that $a, l_1, h_z$, and also we can replace the path $l_1$ such that $b, T_z, h_z, h_z \cdot y_{z-1}, y_{z-1}, l_1, x_{r,j+1}$, $l_2$ is $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $h_z, T_z, a$, and also we can replace the path $l_1$ such that $b_1, T_z, y_{z-1}, y_{z-1}, h_z, h_z, l_2, b, l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

Case 3. $l_1$ is connecting from $a$ to $x_{r,j+1}$, $l_2$ is connecting from $h_z$ to $a_s$, and $l_3$ is connecting from $b_s$ to $b$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $h_z, T_z, h_z, h_z \cdot y_{z-1}, y_{z-1}, l_2, a_s, l_1$ and $l_1$ are still $l_1$ and $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_1$, then we can replace the path $l_2$ such that $a_s, T_z, h_z, h_z \cdot y_{z-1}, y_{z-1}, l_1, x_{r,j+1}$, $l_2$, $b, l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 4. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $h_z, T_z, b_s$, and also we can replace the path $l_1$ such that $a_s, T_z, h_z, h_z \cdot y_{z-1}, y_{z-1}, l_3, b, l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 1. So, the result follows by the induction hypothesis.

Case 4. $l_1$ is connecting from $a$ to $h_z$, $l_2$ is connecting from $a_s$ to $x_{r,j+1}$, and $l_3$ is connecting from $b_s$ to $b$.

In this case, if $y_{z-1} \in l_1$, then we can replace the path $l_1$ such that $a, l_1, y_{z-1}, l_1, y_{z-1}, h_z \cdot T_z, h_z, l_2$ and $l_2$ are still $l_2$ and $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 4. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_2$, then we can replace the path $l_1$ such that $a, l_1, h_z, h_z \cdot y_{z-1}, y_{z-1}, l_2, x_{r,j+1}$ and also we can replace the path $l_2$ such that $h_z, T_z, a_s, l_3$ is
still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ that $h_z$, $l_3$, $b$ and also we can replace the path $l_1$ such that $a$, $l_1$, $h_z$, $h_z \ast y_{z-1}$, $y_{z-1}$, $\bar{T}_z$, $b$, $l_1$ is $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 3. So, the result follows by the induction hypothesis.

Case 5. $l_1$ is connecting from $a$ to $h_z$, $l_2$ is connecting from $a_x$ to $b_y$, and $l_3$ is connecting from $b$ to $x_{r,j+1}$.

In this case, if $y_{z-1} \in l_1$, then we can replace the path $l_1$ such that $a$, $l_1$, $y_{z-1}$, $y_{z-1}$, $h_z$, $h_z$ * $h_z$, $h_z$, $l_2$ and $l_1$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_2$, then we can replace the path $l_1$ such that $a$ * $y_{z-1}$, $y_{z-1}$, $h_z$, $h_z$ * $h_z$, $h_z$, $l_2$ and $l_1$ is still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ that $h_z$, $\bar{T}_z$, $b$ and also we can replace the path $l_1$ such that $a$, $l_1$, $h_z$, $h_z$ * $y_{z-1}$, $y_{z-1}$, $l_3$, $x_{r,j+1}$, $l_1$ is $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 5. So, the result follows by the induction hypothesis.

Case 6. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $h_z$ to $x_{r,j+1}$, and $l_3$ is connecting from $a_x$ to $b$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $h_z$, $\bar{T}_z$, $h_z$, $h_z$ * $y_{z-1}$, $y_{z-1}$, $l_2$, $x_{r,j+1}$, $l_1$ and $l_1$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 6. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $h_z$, $l_1$, $b_y$ and also we can replace the path $l_1$ such that $a$, $l_1$, $y_{z-1}$, $y_{z-1}$, $h_z$, $h_z$, $l_2$, $x_{r,j+1}$, $l_1$ is $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 6. So, the result follows by the induction hypothesis.

Case 7. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $h_z$ to $a_x$, and $l_3$ is connecting from $b$ to $x_{r,j+1}$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $h_z$, $\bar{T}_z$, $h_z$, $h_z$ * $y_{z-1}$, $y_{z-1}$, $l_2$, $a_x$ and $l_1$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 7. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_1$ that $a$, $l_1$, $h_z$ and also we can replace the path $l_2$ such that $a_x$, $\bar{T}_z$, $h_z$, $h_z$ * $y_{z-1}$, $y_{z-1}$, $l_3$, $l_3$, $b_y$, $l_3$ is still $l_3$. So, the result follows by the induction hypothesis.
These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_1$, then we can replace the path $l_2$ such that $h_z$, $\Gamma_z$, $b$ and also we can replace the path $l_3$ such that $a_s$, $\Gamma_s$, $h_z \cdot y_{z-1}$, $y_{z-1}$, $l_1$, $x_{r_{j+1}}$, $l_3$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 8. So, the result follows by the induction hypothesis.

Case 8. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $h_z$ to $b$, and $l_3$ is connecting from $a_s$ to $x_{r_{j+1}}$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $h_z$, $\Gamma_z$, $h_z$, $h_z \cdot y_{z-1}$, $y_{z-1}$, $l_2$, $b$, $l_1$ and $l_3$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 8. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_1$, then we can replace the path $l_3$ such that $b_y$, $\Gamma_y$, $y_{z-1}$, $y_{z-1} \cdot h_z$, $h_z$, $l_2$, $b$, $l_1$ and $l_3$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 9. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$ then we can replace the path $l_2$ such that $h_z$, $\Gamma_z$, $h_z \cdot y_{z-1}$, $y_{z-1}$, $l_2$, $b$, $l_1$ and $l_3$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 9. So, the result follows by the induction hypothesis.

Case 9. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $h_z$ to $b$, and $l_3$ is connecting from $b_y$ to $x_{r_{j+1}}$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $h_z$, $\Gamma_z$, $h_z$, $h_z \cdot y_{z-1}$, $y_{z-1}$, $l_2$, $b$, $l_1$ and $l_3$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 10. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_1$, then we can replace the path $l_3$ such that $a_s$, $\Gamma_s$, $y_{z-1}$, $y_{z-1} \cdot h_z$, $h_z$, $l_2$, $b$, $l_1$ and also we can replace the path $l_3$ such that $a_s$, $l_1$, $y_{z-1} \cdot h_z$, $h_z$, $l_2$, $b$, $l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 11. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ such that $b_y$, $\Gamma_y$, $h_z$, $h_z \cdot y_{z-1}$, $y_{z-1}$, $l_3$, $x_{r_{j+1}}$, $l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 10. So, the result follows by the induction hypothesis.

Case 10. $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $h_z$ to $b_y$, and $l_3$ is connecting from $b_y$ to $x_{r_{j+1}}$.

In this case, if $y_{z-1} \in l_2$, then we can replace the path $l_2$ such that $h_z$, $\Gamma_z$, $h_z$, $h_z \cdot y_{z-1}$, $y_{z-1}$, $l_2$, $b_y$, $l_1$ and $l_3$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 10. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_1$, then we can replace the path $l_2$ such that $a_s$, $\Gamma_s$, $y_{z-1}$, $y_{z-1} \cdot h_z$, $h_z$, $l_2$, $b_y$ and also we can replace the path $l_1$ such that $a_s$, $l_1$, $h_z$. 
$l_1$ is still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_2$ that $h_z$, $\mathcal{T}_1$, $b$ and also we can replace the path $l_1$ such that $b_y$, $\mathcal{T}_1$, $h_z$, $h_z \ast y_{z-1}$, $y_{z-1}$, $l_3$, $x_{r, j+1}$, $l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 9. So, the result follows by the induction hypothesis.

Case 11. $l_1$ is connecting from $a$ to $h_z$, $l_2$ is connecting from $b_y$ to $x_{r, j+1}$, and $l_3$ is connecting from $a_z$ to $b$.

In this case, if $y_{z-1} \in l_1$, then we can replace the path $l_1$ that $a$, $l_1$, $y_{z-1}$, $y_{z-1} \ast h_z$, $h_z$, $\mathcal{T}_1$, $h_z$, $l_2$ and $l_1$ are still $l_2$ and $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 11. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_2$, then we can replace the path $l_1$ such that $a$, $l_1$, $y_{z-1}$, $y_{z-1} \ast h_z$, $h_z$, $\mathcal{T}_1$, $h_z$, $l_2$, $y_{z-1}$ and also we can replace the path $l_2$ such that $h_y$, $\mathcal{T}_1$, $b_y$, $l_3$ is still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 1. So, the result follows by the induction hypothesis.

If $y_{z-1} \in l_3$, then we can replace the path $l_1$ that $a$, $l_1$, $h_z$, $h_z \ast y_{z-1}$, $y_{z-1}$, $l_2$, $x_{r, j+1}$, $a_z$, and also we can replace the path $l_2$ such that $h_y$, $l_1$, $l_1$, $l_3$ is $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $z'$ of Case 9. So, the result follows by the induction hypothesis.

So, Subclaim 4 follows. 

Since $V(P)$ is finite, the sequence of sets $H_0 \subseteq H_1 \subseteq \cdots$ must be constant from some point onwards. Let $H$ be the final sets.

Subclaim 4 implies that there do not exist two distinct vertices $h_i$ and $b_y$ in $P_i$ for $i < i'$. For otherwise, choose $z$ minimal and consider three paths as follows:

(a) $a$, $P$, $b_y$ and $h_z$, $P$, $a_z$ and $x_{r, j+1}$, $P$, $b$.

(b) $a$, $P$, $h_z$ and $b_y$, $P$, $a_z$ and $x_{r, j+1}$, $P$, $b$.

But, by Subclaim 4, such three paths do not exist.

Also, Subclaim 4 implies that if there exist two distinct vertices $h_i$ and $b_y$ in $P_i$ for $i > i''$, then $a$, $P$, $b_y$ is longer than $a$, $P$, $h_z$. For otherwise, choose $z$ minimal, and consider three paths as follows:

(c) $a$, $P$, $a_z$ and $x_{r, j+1}$, $P$, $b_y$ and $h_z$, $P$, $b$.

But, by Subclaim 4, such three paths do not exist.

Subclaim 4 also implies that there do not exist two distinct vertices $h_i$ and $a_z$ in $P_i$ for $i > i''$. For otherwise, choose $z$ minimal and consider three paths as follows:
(d) \( a, P, x_{r,j}, x_{r',k}, x_{r',l}, P, x_{r',j+1} \) and \( b, P, h_i \) and \( a_s, P, b \).

(e) \( a, P, x_{r,j}, x_{r',k}, x_{r',l}, P, x_{r',j+1} \) and \( b, P, a_s, h_i, P, b \).

But, by Subclaim 4, such three paths do not exist.

Now we observe the case that there exist two distinct vertices \( h_e \in H_e \) and \( h_{e} \in H_{e'} \) in \( P_i \) for \( i > l' \). Assume \( h_e \in H_e \) and \( h_{e} \in H_{e'} \) in \( P_i \) for some \( i > l' \) and suppose \( a, P, h_i \) is shorter than \( a, P, h_{e} \). By (c), \( a, P, h_i \) is shorter than \( a, P, b \). We choose \( z \) and \( z' \) minimal. First we observe the vertex \( h_i \) (if \( z > z' \), we assume \( h_i \) as \( h_i \)). If \( z = 0 \), then there must a path \( h \ast h_0 \), where \( h \in U \). But this contradicts the maximality of \( N(x_{r,j}, x_{r,k}) \). So, we may assume \( z > 0 \). We can choose a path \( h_i \ast y_{z-1} \) connecting \( h_i \) to \( y_{z-1} \), where \( y_{z-1} \in \text{Int}(H_{z-1}) \). We may assume \( s \neq i', l' \).

The following statement holds.

(I) \( y_{z-1} \neq P_i, \) for \( s < l' \).

For otherwise, if \( y_{z-1} \in P_i \) for \( s < l' \), then we can choose a vertex \( h_{z-1} \) such that \( h_{z-1}, P, y_{z-1} \) does not contain edges in \( L \). Then we can take three paths \( a, P, h_i \) and \( x_{r,j+1} \), \( P, h_i, h_j, y_{z-1}, P, a_s \) and \( b, P, b \) which satisfy Subclaim 4, a contradiction. So, the result follows.

And also, the following statement holds.

(II) If \( y_{z-1} \in P_i \) for \( s > l' \), then \( a, P, y_{z-1} \) is shorter than \( a, P, h_i \).

For otherwise, if \( y_{z-1} \in P_i \) for \( s > l' \) and \( a, P, y_{z-1} \) is longer than \( a, P, h_i \), then we can choose a vertex \( h_{z-1} \) such that \( h_{z-1}, P, y_{z-1} \) does not contain edges in \( L \). Then we can take three paths \( a, P, a_s \) and \( x_{r,j+1}, P, h_i, h_j, y_{z-1}, P, a_s, b \) and \( b \) which satisfy Subclaim 4, a contradiction. So, the result follows.

Now, we consider the case that \( y_{z-1} \in P_i \) for \( s > l' \) and \( a, P, y_{z-1} \) is shorter than \( a, P, h_i \). Suppose \( z < z' \). Then we can take \( b \in B \) such that \( b, P, y_{z-1} \) does not contain edges in \( L \). In this case, we can take three paths \( a, P, h_i, P, y_{z-1}, P, a_s, y_{z-1}, \tilde{P}, b \) and \( b \) which satisfy Subclaim 4 unless \( h_i \) "comes" from \( \text{Int}(B) \cap V(P_i) \). (The word "come" means that there exists a vertex \( y_{z-1} \in \text{Int}(B) \cap V(P_i) \) such that if we remove the vertex \( y_{z-1} \), then \( h_i \) does not exist.) But in this worst case, either we have three vertex disjoint paths which satisfy Subclaim 4 or both \( h_i \) and \( h_i \) "come" from \( y_{z-1} \) if we choose \( P \) suitably. (We call such vertex "bad.")

The case \( z > z' \) follows from the similar way because we can take three paths \( a, P, x_{r,j+1}, P, y_{z-1}, h_i, P, b \) and \( b \).

So, we may assume that there do not exist two vertices \( h_e \in H_e \) in \( P_i \) for \( i > l' \) or if exist, then there exists a bad vertex. This implies that if we remove all bad vertices, we may assume that there exist no two distinct vertices \( h_e \in H_e \) and \( h_{e} \in H_{e'} \) in \( P_i \) for \( i > l' \).
By the maximality of $N(x_{r,j}, x_{r,k})$ and the fact that we may assume that there do not exist two vertices $h'_i \in H'_i$ and $h''_i \in H''_i$ in $P_i$ for $i < i'$ or there exists a bad vertex, if there exists a vertex $h_i \in H_i$ in $P_i$ for $i > i''$ after deleting all bad vertices, then we can choose a path $h_i \ast y_{z-1}$ connecting $h_i$ to $y_{z-1}$, where $y_{z-1} \in \Int(H_{z-1})$, and $y_{z-1}$ must be in $\Int_r(H_{z-1})$. We consider two cases whether $\Int_r(H_0) = \emptyset$ or not.

Case 1. $\Int_r(H_0) = \emptyset$.

In this case, there does not exist a vertex $h_i \in H_i$ in $P_i$ for $i > i''$. Since $|\Fr(B)| = k-1$, so $|\Fr(H)| \leq k + 1$. Also, by Subclaim 4, we can get the fact that there does not exist two distinct vertices $m'$ and $n'$ in $P_{k-1}$ such that $m' \in H$ and $n' \in A$. Therefore, we can get the fact that $|\Fr(H)| \leq k - 1$.

We claim that $\Fr_r(H) = \{x_{r,j}\}$. For otherwise, if there exists a vertex $h_i \in H_i$ in $P_i$ and $a, P, h_i$ is shorter than $a, P, a, a$, then we can take three paths $a, P, h_i$ and $x_{r,j+1}, P, x_{r,k}, x_{r,k} \ast x_{r,i}, x_{r,i}, P, a, b, P, b$ which satisfy Subclaim 4, and hence the result follows.

Assume there exists a vertex $h_i \in H_i$ in $P_i$ and $a, P, h_i$ is longer than $a, P, a, a$. First, we claim $z > 0$. For otherwise, we can take a path $q \ast h_0$, where $q \in H$. But we can take new path $P'$ extending $P$, that is, $a, P, h_0, h_0 \ast q, q, P, h, a$ and also we can get the fact that $|x_{r,1}, P, h_0|$ is smaller than $|x_{r,1}, P, x_{r,j}|$, which is contrary to the minimality of $|x_{r,1}, P, x_{r,j}|$. So, we may assume $z > 0$. Then we can choose a path $h_i \ast y_{z-1}$ connecting $h_i$ to $y_{z-1}$, where $y_{z-1} \in \Int_r(H_{z-1})$. And also, we can choose a vertex $h_{z-1}$ such that $h_{z-1}, P, y_{z-1}$ does not contain edges in $L$. Note that $i < i'$. In this case, we can take three paths $a, P, y_{z-1}, y_{z-1} \ast h_{z-1}, h_{z-1}, P, x_{r,j}, x_{r,j} \ast x_{r,k}, x_{r,k}, P, x_{r,j+1}$ and $h_{z-1}, P, a, b, P, b$ which satisfy Subclaim 4, and hence the result follows.

So, we may assume that $\Fr_r(H) = \{x_{r,j}\}$ and $|\Fr(H)| \leq k - 2$.

In this case, $\Fr(H) \cup \{x_{r,1}\}$ is a cutset separating $U$ from $a$ and also, $U$ from $h_i$ and its cardinality is at most $k-1$, which is contrary to the connectivity of $G$. Note that the proof of the fact $\Fr_r(H) = \{x_{r,j}\}$ even works when $\Int_r(H_0) \neq \emptyset$ and we delete all bad vertices, since we may assume that there do not exist two vertices $h'_i \in H'_i$ and $h''_i \in H''_i$ in $P_i$ for $i < i'$ or there exists a bad vertex, and hence if we cut all bad vertices, then we do not have to consider the case that there exist two vertices $h'_i \in H'_i$ and $h''_i \in H''_i$ in $P_i$ for $i < i'$.

Case 2. $\Int_r(H_0) \neq \emptyset$.

Let $U'$ be the set of vertices in $\Int_r(H_0)$. Let $H'_0$ be

$$H'_0 := I(U', U')$$

and for $z \geq 1$, let $H'_z$ be

$$H'_z := H'_{z-1} \cup I(\Int(H'_{z-1}), U').$$
Since \( V(P) \) is finite, the sequence of sets \( H_0' \subseteq H_1' \subseteq \cdots \) must be constant from some point onwards. Let \( H' \) be the final sets. Note that \( H' \subseteq H \cup U \).

Since we may assume that there do not exist two vertices \( h'_i \in H'_i \) and \( h'_j \in H'_j \) in \( P_i \) for \( i < i' \) or there exists a bad vertex, so \( |Fr_{r_i}(H')| \leq 1 \) for any \( i > i'' \) if we delete all bad vertices. And also, by the same argument to apply \( U \) to \( Int_{r_i}(H_0) \), we may also assume that there do not exist two vertices \( h'_i \in H'_i \) and \( h'_j \in H'_j \) in \( P_i \) for \( i < i' \) or there exists a bad vertex. Hence \( |Fr_{r_i}(H')| \leq 1 \) for any \( i > i'' \) if we delete all bad vertices. Therefore, we may assume that \( |Fr_{r_i}(H')| \leq 1 \) for any \( i > i'' \) and \( i < i' \). Moreover, by Subclaims 2 and 4, there does not exist a vertex \( h' \in H' \) in \( P_i \) and in \( P_{k-1} \).

In this case, \( Fr(H') \) is cutset separating \( U' \) from \( a \) and also, \( U' \) from \( b \), and its cardinality is at most \( k-1 \), which is contrary to the connectivity of \( G \) when there exists a vertex of \( U' \) which is not bad.

Finally, assume that all vertices in \( U' \) are bad. Note that we may assume that \( Fr_{r_i}(H') = \{x_{r_i,j}\} \) if we delete all bad vertices. If there exists at least one vertex of \( U \) which is not bad, then \( Fr(H) \cup \{x_{r_i,j}, x_{r_{i+1},j}\} \) is a cutset and its cardinality is, since there does not exist a vertex \( h \in H \) in \( P_1 \) and in \( P_{k-1} \), at most \( k-1 \), which is contrary to the connectivity of \( G \).

Suppose all the vertices in \( U \cup U' \) are bad. Take the vertex \( u' \in U \cup U' \) such that the number of \( P_i \) such that \( |Fr_{r_i}(H_i)| = 2 \) and all the vertices of \( H_i \cap V(P) \) comes from \( u' \) is smallest number among them. If there exists a non-bad vertex \( h' \in Int(H_1) \) whose \( Fr_{r_1}(H_1) \) comes from \( u' \), then by the same argument, we have a \( k-1 \) cutset. Hence we may assume that all the vertices in \( Int(H_1) \) whose \( Fr_{r_1}(H_1) \) come from \( u' \) are bad. But in this case, we also have a \( k-1 \) cutset which separates \( u' \) by using the same argument in the proof of the preceding paragraph. So, Claim 3 follows.

Let \( x_{r,m} \) be \( \sup_r(A) \) and let \( x_{r,n} \) be \( \inf_r(B) \), respectively. Let \( r \) be the vertex \( x_{r,m} \) and let \( s \) be the vertex \( x_{r,n+1} \). We define the sequence \( A'_0 \subseteq A'_1 \subseteq \cdots \) and the sequence \( B'_0 \subseteq B'_1 \subseteq \cdots \) of subsets of \( V(P) \) as

\[
A'_0 := I(\{r\}, \{r\}) \cup \{x_{r,m}\},
B'_0 := I(\{s\}, \{s\}) \cup \{x_{r,n}\}
\]

and, for any \( m, n \geq 1 \),

\[
A'_m := A'_{m-1} \cup I(Int(A'_{m-1}), \{r\}),
B'_m := B'_{m-1} \cup I(Int(B'_{m-1}), \{s\}).
\]

\( A_{-1} \) and \( B_{-1} \) will be interpreted as \( \emptyset \).

Suppose \( b_j \in B_i \) and \( a_k \in A_i \) for some \( x, y \geq 0 \). We prove the following Claim.
**Claim 4.** The following statements hold.

1. There do not exist two distinct vertices $a'_i$ and $b_i$ in $P$, such that
   $a'_i \in A_n$ and $b_i \in B_n$, for any $n \geq 0$, for $i = 1, \ldots, k-1$ and for some $y \geq 0$.
2. There do not exist two distinct vertices $a_i$ and $b'_m$ in $P$, such that
   $a_i \in A_n$ and $b'_m \in B'_m$, for any $m \geq 0$, for $i = 1, \ldots, k-1$ and for some $x \geq 0$.
3. In $P(l)$, there does not exist a vertex $a'_i$ such that $a'_i \in A'_n$ for $n \geq 0$.
4. In $P(l)$, there does not exist a vertex $b'_m$ such that $b'_m \in B'_m$ for $m \geq 0$.

**Proof.** Since $a$ and $b$ are symmetric, it is sufficient to consider only (1) and (3). If there exist such vertices $a'_i$ and $b_i$, then choosing $n$ minimal, and considering three paths as follows: If $i < i'$, then

- (a) $a, P, b, a'_i, P, r$ and $a_x, P, b$.
- (b) $a, P, a'_i, b, P, r$ and $a_x, P, b$.

If $i > i'$, then

- (c) $a, P, r$ and $a_x, P, b, a'_i, P, b$.
- (d) $a, P, r$ and $a_x, P, a'_i, b, P, b$.

If there exists a vertex $a'_i$ in $P(l)$, then choose $n$ minimal and consider three paths as follows:

- (e) $a, P, r$ and $a_x, P, x_{r, k}, x_{r, k}^* x_{r, j}, x_{r, j}, P, a'_i$ and $b, P, b$.
- (f) $a, P, r$ and $a'_i, P, x_{r, k}, x_{r, k}^* x_{r, j}, x_{r, j}, P, a_x$ and $b, P, b$.
- (g) $a, P, r$ and $a_x, P, b, b$ and $a'_i, P, b$.
- (h) $a, P, r$ and $a_x, P, a'_i, b, P, b$.

If there exists a vertex $a'_i$ in $x_{r, j+1}, P, x_{r, m}$, then choose $n$ minimal and consider three paths as follows:

- (i) $a, P, r$ and $a'_i, P, x_{r, k}, x_{r, k}^* x_{r, j}, x_{r, j}, \bar{P}, a_x$ and $b, P, b$.

To prove (1) and (3), it is sufficient to prove the following subclaim.

**Subclaim 5.** If there exist three paths $l_1, l_2,$ and $l_3$ in the following cases:

- **Case 1.** $l_1$ is connecting from $a$ to $r$, $l_2$ is connecting from $a_x$ to $a'_i$, and $l_3$ is connecting from $b_x$ to $b$.
- **Case 2.** $l_1$ is connecting from $a$ to $r$, $l_2$ is connecting from $a_x$ to $b$, and $l_3$ is connecting from $a'_i$ to $b$. 


Case 3. \( l_1 \) is connecting from \( a \) to \( b \), \( l_2 \) is connecting from \( a' \) to \( r \), and \( l_3 \) is connecting from \( a_2 \) to \( b \).

Case 4. \( l_1 \) is connecting from \( a \) to \( b \), \( l_2 \) is connecting from \( a' \) to \( a_2 \), and \( l_3 \) is connecting from \( r \) to \( b \).

Case 5. \( l_1 \) is connecting from \( a \) to \( a' \), \( l_2 \) is connecting from \( b \) to \( r \), and \( l_3 \) is connecting from \( a_2 \) to \( b \).

Case 6. \( l_1 \) is connecting from \( a \) to \( a' \), \( l_2 \) is connecting from \( b \) to \( r \), and \( l_3 \) is connecting from \( a_2 \) to \( b \).

Case 7. \( l_1 \) is connecting from \( a \) to \( a' \), \( l_2 \) is connecting from \( a_2 \) to \( r \), and \( l_3 \) is connecting from \( b \) to \( b \).

Case 8. \( l_1 \) is connecting from \( a \) to \( a' \), \( l_2 \) is connecting from \( a_2 \) to \( a_2 \), and \( l_3 \) is connecting from \( b \) to \( r \).

Case 9. \( l_1 \) is connecting from \( a \) to \( a' \), \( l_2 \) is connecting from \( a_2 \) to \( r \), and \( l_3 \) is connecting from \( a_2 \) to \( b \).

And also, the conditions \((S_i)\) below are satisfied:

\((S_1)\) \( l_1 \cup l_2 \cup l_3 \) includes all the edges in \( L \) and all the vertices in \( \text{Int}(B) \) and in \( \text{Int}(A'_{n-1}) \).

\((S_2)\) The only vertices of \( l_1 \cup l_2 \cup l_3 \) not in \( P \) occur in segment of \( l_1 \setminus L \), \( l_2 \setminus L \) and \( l_3 \setminus L \) of the form \( w \ast x \), where \( w \) and \( x \) are both in \( P \) but not both in \( A'_n \).

\((S_3)\) For each of the paths \( l_1 \setminus L, l_2 \setminus L \) and \( l_3 \setminus L \), say \( Q_i \), and each \( n' \leq n-1 \), if there is a vertex \( q \) such that \( q \in Q_i \cap \text{Int}(A'_n) \), then there are two vertices of \( \text{Fr}(A'_n) \) occurring before and after \( q \) along \( Q_i \), and each of the vertices between then along \( Q_i \) is in \( \text{Int}(A'_n) \).

Then, there exist one or two disjoint circuits which contain all the edges in \( L \).

**Proof.** We prove Subclaim 5 by induction on \( n \). Suppose that \( n = 0 \).

Let \( T_1 \) be a path \( T_1 = r \ast a_0 \). \( T_1 \) does not intersect any segment \( w, w \ast x \), \( x \) in \( l_1 \) or in \( l_2 \) or in \( l_3 \) with \( w \) and \( x \) in \( P \) except for its end vertices \( a_0 \) and \( r \). For otherwise, both \( a_0 \) and \( r \) are in \( A'_0 \), which is contrary to \((S_2)\).

Suppose \( x_{r \ast w} \in A' \setminus A_{n-1} \). If \( p > x \) or \( \text{Int}_{r}(A) = \emptyset \), then it is easy to see that \( l_1 \cup l_2 \cup l_3 \) contains all the vertices in \( \text{Int}(A_{n-1}) \). Hence we only consider the case \( r \in \text{Int}(A_{n-1}) \) and \( p \leq x \). This implies \( A' = \bigcup_{i=1}^{j} A'_i \subseteq A \). By the inductive argument, \( l_1 \cup l_2 \cup l_3 \) contains all the vertices in \( \text{Int}(A_{n-1}) \) unless \( a_0 \) “comes” from \( \text{Int}(A_{n-1}) \cap V(P_r) \) and \( a_0 \notin V(P_r) \). (The word “come” means that there exists a vertex \( q \in \text{Int}(A) \cap V(P_r) \) such that if we remove the vertex \( q \), then \( a_0 \) does not exist.) But in this worst case, either we have two vertex disjoint paths which satisfy Claim 1 by using the remark of Claim 1 or by using the inductive argument, or both \( a_0 \) and \( a_0 \) “come” from \( r \) (We call such a vertex “bad.”), or we have a \( k-1 \) cutset.
This implies that we may assume that $l_1 \cup l_2 \cup l_3$ contains all the vertices in Int($A_{x-1}$) otherwise there exists a $k-1$ cutset, which is contrary to the connectivity.

We consider nine cases for $l_1$, $l_2$, and $l_3$.

**Case 1.** $l_1$ is connecting from $a$ to $r$, $l_2$ is connecting from $a_x$ to $a_0'$, and $l_3$ is connecting from $b_y$ to $b$.

In this case, we can get two paths $a_x$, $l_2$, $a_0'$, $a_0' \ast r$, $r$, $T_1$, $a$ and $l_1$ which satisfy Claim 1 and hence, the result follows.

**Case 2.** $l_1$ is connecting from $a$ to $r$, $l_2$ is connecting from $a_x$ to $b_y$, and $l_3$ is connecting from $a_0'$ to $b$.

In this case, we can get two paths $a$, $l_1$, $r$, $r \ast a_0'$, $a_0'$, $l_3$, $b$ and $l_2$ which satisfy Claim 1 and hence, the result follows.

**Case 3.** $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $a_0'$ to $r$, and $l_3$ is connecting from $a_x$ to $b$.

In this case, we can get a cycle $a_0'$, $l_2$, $r$, $r \ast a_0'$, $a_0'$ and two paths $l_1$ and $l_3$ which satisfy Subclaim 1, and hence, the result follows.

**Case 4.** $l_1$ is connecting from $a$ to $b_y$, $l_2$ is connecting from $a_0'$ to $a_x$, and $l_3$ is connecting from $r$ to $b$.

In this case, we can get two paths $a_x$, $l_2$, $a_0'$, $a_0' \ast r$, $r$, $l_3$, $b$ and $l_1$ which satisfy Claim 1, and hence, the result follows.

**Case 5.** $l_1$ is connecting from $a$ to $a_0'$, $l_2$ is connecting from $b_y$ to $r$, and $l_3$ is connecting from $a_x$ to $b$.

In this case, we can get two paths $a$, $l_1$, $a_0'$, $a_0' \ast r$, $r$, $T_2$, $b_y$ and $l_3$ which satisfy Claim 1, and hence, the result follows.

**Case 6.** $l_1$ is connecting from $a$ to $a_0'$, $l_2$ is connecting from $b$ to $r$, and $l_3$ is connecting from $a_x$ to $b_y$.

In this case, we can get two paths $a$, $l_1$, $a_0'$, $a_0' \ast r$, $r$, $T_2$, $b$ and $l_3$ which satisfy Claim 1, and hence, the result follows.

**Case 7.** $l_1$ is connecting from $a$ to $a_0'$, $l_2$ is connecting from $a_x$ to $r$, and $l_3$ is connecting from $b_y$ to $b$.

In this case, we can get two paths $a$, $l_1$, $a_0'$, $a_0' \ast r$, $r$, $T_2$, $a_x$ and $l_3$ which satisfy Claim 1 and hence, the result follows.

**Case 8.** $l_1$ is connecting from $a$ to $b$, $l_2$ is connecting from $a_x$ to $a_0'$, and $l_3$ is connecting from $b_y$ to $r$. 
In this case, we can get two paths $a_1, l_2, a'_0, a'_0 \ast r, r, T_3, b$, and $l_1$ which satisfy Claim 1, and hence, the result follows.

**Case 9.** $l_1$ is connecting from $a$ to $b$, $l_2$ is connecting from $a_1$ to $r$, and $l_3$ is connecting from $a'_0$ to $b$. 

In this case, we can get two paths $b, T_3, a_1, a'_0, a'_0 \ast r, r, T_2, a$, and $l_1$ which satisfy Claim 1, and hence, the result follows.

Suppose $n > 0$. We may assume $a'_n \in \mathcal{A}_n \setminus \mathcal{A}_{n-1}$. For otherwise, the result follows by the induction hypothesis. We can choose a path $a'_n \ast y_{n-1}$ connecting $a'_n$ to $y_{n-1}$, where $y_{n-1} \in \text{Int}(\mathcal{A}_{n-1})$. This path does not intersect any segment $w, w \ast x, x$ in $l_1$ or in $l_2$ or in $l_3$ with $w$ and $x$ in $P$ except for its end vertices $a'_n$ and $y_{n-1}$. For otherwise, both $a'_n$ and $y_{n-1}$ are in $\mathcal{A}_n$, which is contrary to (S3). By the condition (S3), there exists a vertex $a'_{n-1} \in \mathcal{A}_{n-1}$ which is preceding $y_{n-1}$ such that the segment $a'_{n-1}, l_1, y_{n-1}$ or $a'_{n-1}, l_2, y_{n-1}$ or $a'_{n-1}, l_3, y_{n-1}$ does not contain edges in $L$. We choose a vertex $a'_{n'}$ which is the last vertex before $y_{n-1}$ along $l_1$ (if $y_{n-1} \in l_1$) or along $l_2$ (if $y_{n-1} \in l_2$) or along $l_3$ (if $y_{n-1} \in l_3$) such that the segment $y_{n-1}, l_1, a'_{n-1}$ or $y_{n-1}, l_2, a'_{n-1}$ or $y_{n-1}, l_3, a'_{n-1}$ does not contain edges in $L$. We choose a vertex $a''_{n'}$ which is the last vertex after $y_{n-1}$ along $l_1$ (if $y_{n-1} \in l_1$) or along $l_2$ (if $y_{n-1} \in l_2$) or along $l_3$ (if $y_{n-1} \in l_3$) such that $a''_{n'} \in \text{Fr}(\mathcal{A}'_{n'})$ for any $n' \leq n-1$, and choose $n'$ minimal so that $a''_{n'} \notin \text{Cl}(\mathcal{A}'_{n-1})$. Also by the condition (S3), there exists a vertex $a''_{n-1} \in \mathcal{A}_{n-1}$ which is succeeding $y_{n-1}$ such that the segment $y_{n-1}, l_1, a''_{n-1}$ or $y_{n-1}, l_2, a''_{n-1}$ or $y_{n-1}, l_3, a''_{n-1}$ does not contain edges in $L$. We choose a vertex $a''_{n'}$ which is the last vertex after $y_{n-1}$ along $l_1$ (if $y_{n-1} \in l_1$) or along $l_2$ (if $y_{n-1} \in l_2$) or along $l_3$ (if $y_{n-1} \in l_3$) such that $a''_{n'} \in \text{Fr}(\mathcal{A}'_{n'})$ for any $n' \leq n-1$, and choose $n'$ minimal so that $a''_{n'} \notin \text{Cl}(\mathcal{A}'_{n-1})$. We will write $a''_n$ instead of $a'_{n'}$ since it may not be confusing for readers.

Then there does not exist a vertex that is in $\text{Int}(\mathcal{A}'_{n'})$ in the segments both $y_{n-1}, l_1, a''_n$ and $a''_n, l_1, y_{n-1}$ (if $y_{n-1} \in l_1$), or both $y_{n-1}, l_2, a''_n$ and $a''_n, l_2, y_{n-1}$ (if $y_{n-1} \in l_2$), or both $y_{n-1}, l_3, a''_n$ and $a''_n, l_3, y_{n-1}$ (if $y_{n-1} \in l_3$).

We may assume that there are no vertices in $\text{Int}(B)$ in the segments both $y_{n-1}, l_1, a''_n$ and $a''_n, l_1, y_{n-1}$ (if $y_{n-1} \in l_1$), or both $y_{n-1}, l_2, a''_n$ and $a''_n, l_2, y_{n-1}$ (if $y_{n-1} \in l_2$), or both $y_{n-1}, l_3, a''_n$ and $a''_n, l_3, y_{n-1}$ (if $y_{n-1} \in l_3$). For otherwise, there exist some $P_i$ of $P$ which contains distinct vertices $r, w$ such that $r \in \mathcal{A}_n'$ and $w \in A \cup B$. But, in this case, we can take three paths which satisfy the case $n'$. Hence the result follows by the induction hypothesis.

We consider nine cases of $l_1, l_2$, and $l_3$.

**Case 1.** $l_3$ is connecting from $a$ to $r$, $l_2$ is connecting from $a_1$ to $a''_n$, and $l_1$ is connecting from $b$ to $b$. 

In this case, if $y_{n-1} \in l_1$, then we can replace the path $l_2$ such that $a_1, l_2, y_{n-1}, a''_n \ast r, r, T_3, b$, and $l_1$ are still $l_1$ and $l_1$. These three paths $l_1, l_2$, and $l_3$ satisfy the case $n'$ of Case 1. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_1$, then we can replace the path $l_2$ such that $a_1, l_2, l_3, a''_n \ast y_{n-1}, y_{n-1}, l_1, r$ and also we can replace the path $l_1$ such that $a, l_1, a''_n \ast l_3$ is
still \( l_3 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 7. So, the result follows by the induction hypothesis.

If \( y_{n-1} \in l_3 \), then we can replace the path \( l_3 \) such that \( a'_n, l_3, b \) and also we can replace the path \( l_2 \) such that \( a, l_2, a'_n, a'_n \ast y_{n-1}, y_{n-1}, l_1, b \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 2. So, the result follows by the induction hypothesis.

**Case 2.** \( l_1 \) is connecting from \( a \) to \( r \), \( l_2 \) is connecting from \( a'_x \) to \( b_y \), and \( l_3 \) is connecting from \( a'_n \) to \( b \).

In this case, if \( y_{n-1} \in l_3 \), then we can replace the path \( l_3 \) such that \( a'_n, l_3, a'_n, y_{n-1}, y_{n-1}, y_{n-1}, l_1, b \) and \( l_2 \) are still \( l_1 \) and \( l_2 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 2. So, the result follows by the induction hypothesis.

If \( y_{n-1} \in l_2 \), then we can replace the path \( l_2 \) such that \( b, l_2, a'_n, a'_n, y_{n-1}, y_{n-1}, l_1, r \) and also we can replace the path \( l_1 \) such that \( a, l_1, a'_n \ast y_{n-1}, l_1 \ast a'_n \ast y_{n-1} \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 6. So, the result follows by the induction hypothesis.

If \( y_{n-1} \in l_2 \), then we can replace the path \( l_2 \) such that \( a, l_2, a'_n \) and also we can replace the path \( l_1 \) such that \( b, l_1, a'_n, a'_n, y_{n-1}, l_1, b \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 1. So, the result follows by the induction hypothesis.

**Case 3.** \( l_1 \) is connecting from \( a \) to \( b_y \), \( l_2 \) is connecting from \( a'_x \) to \( r \), and \( l_3 \) is connecting from \( a'_n \) to \( b \).

In this case, if \( y_{n-1} \in l_2 \), then we can replace the path \( l_2 \) such that \( a'_n, l_2, a'_n, y_{n-1}, y_{n-1}, l_1, r \) and also we can replace the path \( l_1 \) such that \( a, l_1, a'_n \ast y_{n-1}, y_{n-1}, y_{n-1}, l_1 \ast a'_n \ast y_{n-1} \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 3. So, the result follows by the induction hypothesis.

If \( y_{n-1} \in l_1 \), then we can replace the path \( l_1 \) such that \( b, l_1, y_{n-1}, y_{n-1}, l_1, r \) and also we can replace the path \( l_1 \) such that \( a, l_1, a'_n, a'_n \ast y_{n-1}, l_1, y_{n-1} \ast a'_n \ast y_{n-1} \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 5. So, the result follows by the induction hypothesis.

If \( y_{n-1} \in l_3 \), then we can replace the path \( l_3 \) such that \( a'_n, l_3, a'_n, y_{n-1}, y_{n-1}, l_1, b \) and also we can replace the path \( l_1 \) such that \( r, l_1, y_{n-1}, y_{n-1}, b \) is still \( l_1 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 4. So, the result follows by the induction hypothesis.

**Case 4.** \( l_1 \) is connecting from \( a \) to \( b_y \), \( l_2 \) is connecting from \( a'_x \) to \( a_x \), and \( l_3 \) is connecting from \( r \) to \( b \).

In this case, if \( y_{n-1} \in l_2 \), then we can replace the path \( l_2 \) such that \( a'_n, l_2, a'_n \ast y_{n-1}, y_{n-1}, l_1, b \) and \( l_2 \) are still \( l_1 \) and \( l_3 \). These three paths \( l_1, l_2, \) and \( l_3 \) satisfy the case \( n' \) of Case 4. So, the result follows by the induction hypothesis.

If \( y_{n-1} \in l_1 \), then we can replace the path \( l_1 \) such that \( a, l_1, a'_n, a'_n \ast y_{n-1}, y_{n-1}, b \) and also we can replace the path \( l_1 \) such that \( a, l_1, a'_n \ast y_{n-1}, y_{n-1}, l_1, b \) is still \( l_1 \).
These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 6. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_3$, then we can replace the path $l_2$ such that $a''_n$, $\overline{T}_n$, $r$ and also we can replace the path $l_1$ such that $a'_n$, $\overline{T}_n$, $a''_n \cdot y_{n-1}$, $y_{n-1}$, $l_3$, $b$. $l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 3. So, the result follows by the induction hypothesis.

**Case 5.** $l_1$ is connecting from $a$ to $a'_n$, $l_2$ is connecting from $b$, to $r$, and $l_3$ is connecting from $a_\gamma$ to $b_\gamma$.

In this case, if $y_{n-1} \in l_1$, then we can replace the path $l_1$ such that $a_n$, $l_1$, $y_{n-1}$, $y_{n-1}$, $a''_n$, $\overline{T}_n$, $a''_n \cdot l_2$ and $l_3$ are still $l_2$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_2$, then we can replace the path $l_1$ such that $a_n$, $l_1$, $a''_n$, $a''_n \cdot y_{n-1}$, $y_{n-1}$, $l_2$, $b_\gamma$ and also we can replace the path $l_2$ such that $a''_n$, $l_2$, $r$, $l_3$ is still $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 3. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_3$, then we can replace the path $l_2$ such that $a''_n$, $l_1$, $a''_n$ and also we can replace the path $l_1$ such that $a_n$, $l_1$, $a''_n$, $a''_n \cdot y_{n-1}$, $y_{n-1}$, $l_1$, $b$. $l_3$ is $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 8. So, the result follows by the induction hypothesis.

**Case 6.** $l_1$ is connecting from $a$ to $a'_n$, $l_2$ is connecting from $b$ to $r$, and $l_3$ is connecting from $a_\gamma$ to $b_\gamma$.

In this case, if $y_{n-1} \in l_1$, then we can replace the path $l_1$ such that $a_n$, $l_1$, $y_{n-1}$, $y_{n-1}$, $a''_n$, $\overline{T}_n$, $a''_n \cdot l_2$ and $l_3$ are still $l_2$ and $l_3$. These three paths $l_1$, $l_2$, $l_3$ satisfy the case $n'$ of Case 6. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_2$, then we can replace the path $l_1$ such that $a_n$, $l_1$, $a''_n$, $a''_n \cdot y_{n-1}$, $y_{n-1}$, $l_2$, $r$ and also we can replace the path $l_1$ such that $a_n$, $l_1$, $a''_n$, $a''_n \cdot l_2$ is $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 2. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_3$, then we can replace the path $l_2$ such that $a''_n$, $\overline{T}_n$, $a_\gamma$ and also we can replace the path $l_1$ such that $a_n$, $l_1$, $a''_n$, $a''_n \cdot y_{n-1}$, $y_{n-1}$, $l_3$, $b_\gamma$, $l_3$ is $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 4. So, the result follows by the induction hypothesis.

**Case 7.** $l_1$ is connecting from $a$ to $a'_n$, $l_2$ is connecting from $a_\gamma$ to $r$ and $l_3$ is connecting from $b_\gamma$ to $b$.

In this case, if $y_{n-1} \in l_1$, then we can replace the path $l_1$ such that $a_n$, $l_1$, $y_{n-1}$, $y_{n-1}$, $a''_n$, $\overline{T}_n$, $a''_n \cdot l_2$ and $l_3$ are still $l_2$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 7. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_2$, then we can replace the path $l_1$ such that $a_n$, $l_1$, $a''_n$, $a''_n \cdot y_{n-1}$, $y_{n-1}$, $l_2$, $r$ and also we can replace the path $l_2$ such that $a_n$, $l_2$, $a''_n \cdot l_3$ is $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 8. So, the result follows by the induction hypothesis.
These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 1. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_3$, then we can replace the path $l_1$ such that $a'_r$, $\bar{T}_3$, $b_y$ and also we can replace the path $l_1$ such that $a$, $l_1$, $a'_r$, $a'_n$, $\bar{T}_2$, $a'_{y-1}$, $l_1$, $b$, $l_2$ is still $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 9. So, the result follows by the induction hypothesis.

**Case 8.** $l_1$ is connecting from $a$ to $b$, $l_2$ is connecting from $a_r$ to $a'_r$ and $l_3$ is connecting from $b_y$ to $r$.

In this case, if $y_{n-1} \in l_2$, then we can replace the path $l_2$ such that $a$, $l_2$, $y_{n-1}$, $a'_r$, $a'_n$, $\bar{T}_3$, $a'_y$, $l_1$, $l_3$ and $l_3$ are still $l_1$ and $l_3$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 8. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_3$, then we can replace the path $l_3$ such that $a'_r$, $\bar{T}_3$, $b_y$ and also we can replace the path $l_3$ such that $a$, $l_1$, $a'_r$, $l_2$ is $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 5. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_3$, then we can replace the path $l_3$ such that $a'_r$, $\bar{T}_3$, $b_y$ and also we can replace the path $l_3$ such that $a$, $l_1$, $a'_r$, $a'_n$, $\bar{T}_3$, $a'_y$, $l_1$, $l_3$, $r$, $l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 9. So, the result follows by the induction hypothesis.

**Case 9.** $l_1$ is connecting from $a$ to $b$, $l_2$ is connecting from $a_r$ to $r$, and $l_3$ is connecting from $a'_r$ to $b_y$.

In this case, if $y_{n-1} \in l_3$, then we can replace the path $l_3$ such that $a'_r$, $\bar{T}_3$, $a'_n$, $\bar{T}_3$, $a'_r$, $y_{n-1}$, $l_3$, $b_y$, $l_1$ and $l_2$ are still $l_1$ and $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 9. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_1$, then we can replace the path $l_1$ such that $b_y$, $\bar{T}_3$, $a'_r$, $a'_n$, $\bar{T}_3$, $a'_y$, $y_{n-1}$, $l_1$, $b$ and also we can replace the path $l_1$ such that $a$, $l_1$, $a'_r$, $l_2$ is still $l_2$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 7. So, the result follows by the induction hypothesis.

If $y_{n-1} \in l_2$, then we can replace the path $l_2$ such that $a$, $l_2$, $a'_n$ and also we can replace the path $l_2$ such that $b_y$, $\bar{T}_3$, $a'_r$, $a'_n$, $\bar{T}_3$, $a'_y$, $y_{n-1}$, $l_1$, $r$, $l_1$ is still $l_1$. These three paths $l_1$, $l_2$, and $l_3$ satisfy the case $n'$ of Case 8. So, the result follows by the induction hypothesis.

So, Subclaim 5 follows. Therefore, Claim 4 follows.

Since $V(P)$ is finite, the sequence of sets $A'_0 \subseteq A'_1 \subseteq \cdots$ and the sequence $B'_0 \subseteq B'_1 \subseteq \cdots$ must be constant from some point onwards. Let $A'$ and $B'$ be the final sets.

By Claims 3 and 4, we can get the fact that either $|Fr(A')| \leq k-3$ or $|Fr(B')| \leq k-3$. Without loss of generality, we may assume that $|Fr(A')| \leq k-3$. If $r \in Int(A')$, then $Fr(A') \cup \{a\}$ is a cutset separating $r$
from $b$ and its cardinality is at most $k-2$, which is contrary to the connectivity of $G$.

If \( r \not\in \text{Int}(A') \), then \( x_{r,m} \) is in \( \text{Fr}(A') \). In this case, \( \text{Fr}(A') \cup \{a\} \cup \{x_{r,m-2}\} \) is a cutset separating $r$ from $b$ and its cardinality is at most $k-1$, which is contrary to the connectivity of $G$. So Theorem 2 follows.

5. OUTLINE OF THE PROOF OF THE LOVÁSZ–WOODALL CONJECTURE

Woodall [18] also proved the following.

**Theorem 3.** If $L$ is a set of $k$ independent edges in a $(k+1)$-connected graph $G$, and $G - \{a, b\}$ has a circuit containing all the edges of $L \setminus \{(a, b)\}$, where $(a, b) \in L$, then $G$ has a circuit containing all the edges of $L$.

The author [10] proved the following.

**Theorem 4.** Let $L$ be a set of $k$ independent edges in a $k$-connected graph $G$, let $e$ be an edge in $L$, and define $L' := L \setminus e$. If there exist two disjoint circuits $C_1$ and $C_2$ such that $C_1$ contains $e$ and $C_2$ contains $L'$, then $G$ contains a circuit that contains all the edges in $L$.

To compare Theorem 3 with Woodall’s result, the assumption that there exists a circuit $C$ which contains all the edges in $L' := L \setminus e$, where $e$ is one of $L$, is in common. And also, the assumption that $V(C) \cap V(e) = \emptyset$ is in common. But, if there exists a circuit in $G \setminus C$ which contains $e$, then the connectivity drops from $k+1$ to $k$.

By Theorem 4, there exist one or two disjoint circuits that contain all the edges in $L$. If there exists one circuit that contains all the edges in $L$, then Conjecture 1 holds. So we may assume that there exist two disjoint circuits $C_1$ and $C_2$ such that $C_1$ contains $L'$ and $C_2$ contains $L''$, where $L' \cup L'' = L$ and $|L'| \leq |L''|$. We consider the induction on $|L'|$. By Theorem 4, if $|L'| = 1$, then Conjecture 1 holds.

Theorem 4 is the first step toward Lovász–Woodall Conjecture. In addition, we get the following theorem in [10] by using Theorem 4.

**Theorem 5.** Let $L$ be a set of $k$ independent edges in a $k$-connected graph $G$, let $e_1, e_2$ be two edges in $L$, and define $L' := L \setminus \{e_1, e_2\}$. If there exist two disjoint circuits $C_1$ and $C_2$ such that $C_1$ contains $e_1$ and $e_2$, and $C_2$ contains $L'$, then $G$ contains a circuit that contains all the edges in $L$.

We also get the following theorem in [10] by using Theorems 4 and 5.
Theorem 6. Let \( L \) be a set of \( k \) independent edges in a \( k \)-connected graph \( G \), let \( e_1, e_2, e_3 \) be three edges in \( L \), and define \( L' := L \setminus \{e_1, e_2, e_3\} \). If there exist two disjoint circuits \( C_1 \) and \( C_2 \) such that \( C_1 \) contains \( e_1, e_2, \) and \( e_3 \), and \( C_2 \) contains \( L' \), then \( G \) contains a circuit that contains all the edges in \( L \).

By using Theorems 2, 4, and 5, we get the following corollaries which imply the results of Erdős and Győri [4], Lomonosov [13], and Sanders [16].

Corollary 7. Let \( L \) be a set of 4 independent edges in a 4-connected graph \( G \). Then \( G \) has a circuit containing all the edges of \( L \).

Corollary 8. Let \( L \) be a set of 5 independent edges in a 5-connected graph \( G \). If \( G - L \) is connected, then \( G \) has a circuit containing all the edges of \( L \).

By using Theorems 2, 4, 5, and 6, we also get the following corollaries which have not yet been known.

Corollary 9. Let \( L \) be a set of 6 independent edges in a 6-connected graph \( G \). Then \( G \) has a circuit containing all the edges of \( L \).

Corollary 10. Let \( L \) be a set of 7 independent edges in a 7-connected graph \( G \). If \( G - L \) is connected, then \( G \) has a circuit containing all the edges of \( L \).

Now we turn back to the proof of Conjecture 1. Our main tool is to consider the induction on \(|L'|\). In [11], we prove the following theorem.

Theorem 11. Let \( L \) be a set of \( k \) independent edges in a \( k \)-connected graph \( G \). If there exist two disjoint circuits \( C_1 \) and \( C_2 \) such that \( C_1 \) contains \( k' \) edges in \( L \) and \( C_2 \) contains \( k'' \) edges in \( L \), where \( k' + k'' = k \) (this implies \( C_1 \cup C_2 \) contains all the edges in \( L \)), then one of the followings holds.

1. \( G \) has a circuit containing all edges in \( L \).
2. There exist two disjoint circuits \( C'_1 \) and \( C'_2 \) such that \( C'_1 \) contains \( k_1 \) edges in \( L \) and \( C'_2 \) contains \( k_2 \) edges in \( L \) (this implies \( C_1 \cup C_2 \) contains all the edges in \( L \)), where \( k_1 + k_2 = k \) and \( k_1 < k', k_2 > k'' \).
3. We can choose \( C_1 \) and \( C_2 \) such that, for any \( v \in G - C_1 - C_2 \), \( V(C_2) \) is cutset separating from \( v \) to \( C_1 \).

This theorem takes a crucial roles in the proof of Lovász–Woodall Conjecture and this theorem is “Key” idea. If (2) holds, then by the induction hypothesis, we can get the result. Note that (3) implies that there do
not exist paths of length at least 2 connecting $C_1$ and $C_2$. There exist only edges connecting $C_1$ and $C_2$. Finally, we prove Conjecture 1 by using Theorems 4, 5, and 11 in [12].

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