Hey Bill, what’s the deal with Induced poset saturation?

Let $S_n$ denote the set of permutations on $n$ symbols. For $k > 0$ and $\pi \in S_n$, $\sigma \in S_{n+k}$ we say that $\pi$ is covered by $\sigma$ if we can delete $k$ symbols and, after reducing what’s left to the alphabet $[n]$ we have a copy of $\pi$. An example of $\pi \in S_3$ and $\sigma \in S_8$ where $\sigma$ covers $\pi$ is:

$$\pi = 132 \text{ is covered by } \sigma = 14286735$$

because the bolded symbols in $\sigma$ have a 132 pattern. That is, $\sigma$ contains $\pi$ as an order isomorphic subpattern. Let $C_{n,n+k}$ denote the minimum size of $A \subseteq S_{n+k}$ so that every member of $S_n$ is covered by some member of $A$. We have the following:

**Theorem 1** (Godbole, K., Lan, Laubmeier, Yuan).

$$k! \frac{(n+k)!}{n^{2k}}(1 + o(1)) \leq C_{n,n+k} \leq k \frac{(n+k)!}{n^{2k}} \log n (1 + o(1)).$$

Notice that the lower bound matches the upper bound up to a logarithmic factor. It remains open to see if the log factor can be removed.

Even more generally, let $C^{(\lambda)}_{n,n+k}$ denote the minimum size of $A \subseteq S_{n+k}$ so that every member of $S_n$ is covered by at least $\lambda$ members of $A$. We have the following:

**Theorem 2** (Godbole, K., Lan, Laubmeier, Yuan).

$$k! \frac{(n+k)!}{n^{2k}}(1 + o(1)) \leq C^{(\lambda)}_{n,n+k} \leq \frac{(n+k)!}{n^{2k}}(k \log n - (\lambda - 1) \log(k \log n) + \frac{\lambda}{(\lambda - 1)!}(1 + o(1))).$$

We also showed the following:

**Theorem 3** (Godbole, K., Lan, Laubmeier, Yuan). Let $A$ be a random subset of $S_{n+k}$ where each element of $S_{n+k}$ is selected for membership in $A$ with probability $p$. Then we have:

$$p \ll \frac{\log n}{n^{2k-1}} \Rightarrow \mathbb{P}(A \text{ covers } S_n) \to 0(n \to \infty)$$

$$p \gg \frac{\log n}{n^{2k-1}} \Rightarrow \mathbb{P}(A \text{ covers } S_n) \to 1(n \to \infty)$$

establishing a probabilistic threshold for permutation covers via Janson’s inequality. Each of these results generalize the previous work of Allison, Godbole, Hawley, and K.