Hey Bill, what’s the deal with *The minimum number of edges in uniform hypergraphs with Property O*?

Given a graph $G$, an orientation of $G$ is a graph obtained by taking each edge and putting an order on it (that is, turning the edges which are sets into 2-tuples). Similarly, one can take a $k$-uniform hypergraph (that is, and graph whose edges are $k$-sets) and orient it by turning the edges into $k$-tuples. Any graph obtained this way is an **oriented $k$-graph**. Let $H = (V, E)$ be an oriented $k$-graph. For a total order $<$ on $V$ and an edge $(x_1, x_2, \ldots, x_k) = e \in E$ we say that $e$ is consistent with $<$ if $x_1 < x_2 < \ldots < x_k$. That is, the tuple $e$ puts the vertices $x_1, x_2, \ldots, x_k$ in the same order as $<$. We say that $H$ has **Property O** if for any total ordering $<$ of $V$, there exists an edge $e \in E$ which is consistent with $<$. Let $f(k)$ denote the minimum number of edges in an oriented $k$-graph which has Property O. The primary aim of this document is to provide bounds on $f(k)$.

As a warm up, here I will include a trivial lower bound on $f(k)$. One proves a lower bound on $f(k)$ by showing that any oriented $k$ graph on few edges must necessarily fail to have Property O. Let $H = (V, E)$ be an arbitrary oriented $k$-graph. For a fixed edge $(x_1, x_2, \ldots, x_k) = e \in E$, we ask “what proportion of linear orders $<$ on $V$ are consistent with $e$?”. One observation is that for $<$ to be consistent with $e$, we only have to consider what the order $<$ does to the elements of $e$. Further, of the $k!$ ways to arrange the elements of $e$, only 1 puts them in the order prescribed by $e$. Hence, only a $\frac{1}{k!}$ proportion of total orders on $V$ are consistent with $e$. Hence, at most $\frac{|E|}{k!}$ total orders are consistent with some edge $e \in E$. To say it another way, when $|E| < k!$, fewer than 100% of the total orders on $V$ are consistent with some edge. That is, there is some total order on $V$ which is not consistent with some edge, and so $H$ fails to have Property O. Thus we have shown:

$$k! \leq f(k)$$

This simple lower bound is the best lower bound that we have. To prove an upper bound, we take a complete $k$-graph and orient the edges independently and randomly. We then figure out for which values of $n$ the probability that a $k$-graph generated this way fails to have Property O dips below 1. For this probability to be less than 1 means that some $k$-graph generated this way must have Property O. Since $f(k)$ is a statement about the number of edges in an oriented $k$-graph with Property O, we provide estimates for $\binom{n}{k}$ for our choice of $n$. This gives the theorem:

**Theorem 1** (Duffus, K. Rödl).

$$k! \leq f(k) \leq (k^2 \ln k)k!$$

What we would like to know is whether $\frac{f(k)}{k!}$ tends to infinity with $k$. Instead, we focus on when a randomly oriented complete $k$-graph typically has Property O. The actual theorem statement is fairly technical, but here is a version with some of the details swept under the rug:

**Theorem 2.** There exists a choice of $n$ and a function $\varepsilon(k) \to 0$ so that a complete oriented $k$ graph on $n$ vertices fails to have Property O with high probability. For this choice of $n$, we have:

$$\binom{n}{k} = k^{1/2-\varepsilon(k)}k!$$

This is the “with high probability” statement for randomly oriented complete $k$-graphs.