Math 211 Final Review

Things to know:
Multivariable functions and their graphs
Vectors
Dot product and projection
Cross product
Partial derivatives
Tangent plane to a graph
Directional derivatives and what they represent
Equality of mixed partial derivatives
Computation of double/triple integrals
Change of variables to polar, cylindrical, spherical coordinates
General change of variables and the Jacobian
Parametrization of curves and surfaces
Vector fields
Line integrals
Fundamental theorem of line integrals and gradient vector fields
Green’s theorem
Surface integrals
Divergence theorem
Stokes’ theorem
Problems to think about:

Problem 0.1. (True/False, 2 pt each) Mark your answers True (T) or False (F). No explanation required!

i: T F. For vectors, \( \vec{v}, \vec{w}, \) and \( \vec{u} \in \mathbb{R}^3 \), \( \vec{v} \times (\vec{w} \cdot (\vec{u} - \vec{v})) \) is a scalar.

ii: T F. \( \vec{F} = \langle yz^2, xz^2 - 3z, 2xyz - 3 \rangle \) is a gradient vector field.

iii: T F. Given two non-zero vectors \( \vec{u}, \vec{v} \) in \( \mathbb{R}^3 \), \( \vec{u} \times (3\vec{v} + \vec{u}) = 0 \) implies that \( \vec{u} \) and \( \vec{v} \) are parallel.

iv: T F. The vector fields \( \vec{F} = \langle x, y, z \rangle \) and \( \vec{G} = \langle -y, z, -x \rangle \) are never perpendicular.

v: T F. For any two vectors, \( \vec{v}, \vec{w} \in \mathbb{R}^3 \), \( \vec{v} \cdot \vec{w} \geq 0 \).

vi: T F. If \( \int_S \vec{F} \cdot d\vec{A} = 0 \) then \( \vec{F} \) is normal to the surface \( S \) at every point.

vii: T F. Fubini’s theorem implies that 
\[
\int_0^1 \int_{x^2}^{3-2x} x^2 \, dy \, dx = \int_0^1 \int_{x^2}^{3-y} x^2 \, dx \, dy.
\]

viii: T F. Green’s Theorem can be used to compute the line integral of \( \vec{F} = \langle \frac{x}{x^2+y^2-1}, \frac{y}{x^2+y^2-1} \rangle \) along the circle of radius 1 centered at \((3,0)\) oriented counter-clockwise.

ix: T F. If \( \vec{F} \) and \( \vec{G} \) are vector fields in \( \mathbb{R}^3 \) such that \( \text{curl}(\vec{F}) = \text{curl}(\vec{G}) \), then \( \vec{F} = \vec{G} \).

x: T F. For a surface given by \( z = f(x, y) \), the normal vector to the tangent plane is \( \{f_x, f_y, 1\} \).

Problem 0.2. (9 pts)

i. If \( \vec{v} \) and \( \vec{w} \) are two perpendicular unit vectors in \( \mathbb{R}^3 \), compute \( ||(\vec{v} + 3\vec{w}) \times (2\vec{v} - \vec{w})|| \).

ii. Consider the surface \( z = xy \). Find all of the points on this surface where the tangent plane is parallel to the plane \( 2x + 2y + 1 = z \).

iii. Let \( z = x^3 + xy^2 \). Find a vector tangent to the contour through the point \((1, -1)\).

Problem 0.3. (8 pts) Consider
\[
\int \int_R y \, dx \, dy
\]
where \( R \) is the region in the first quadrant bounded between \( y = 1/x \), \( y = 3/x \), \( y = x \), and \( y = 3x \). Change variables in this integral from Cartesian coordinates to \( st \)-coordinates: \( x = s/t \), \( y = t \). Do not evaluate!

Problem 0.4. (10 pts) Compute the line integral of \( \vec{F} \) along the curve \( C \) where

i. \( \vec{F} = \langle x - y, x + y \rangle \) and \( C \) is the line segment from \((0,0)\) to \((1,1)\) followed by the line segment from \((1,1)\) to \((0,1)\).
ii. \( \vec{F} = \langle 2xe^y, x^2e^y + e^y + ye^y \rangle \) and \( C \) is the top half of a circle of radius 1 centered at the origin, oriented clockwise.

**Problem 0.5.** (10 pts) Compute the flux of \( \vec{F} \) through the surface \( S \) where

i. \( \vec{F} = \langle y, z, x \rangle \) and \( S \) is the triangular region with vertices at \((2,0,0), (0,2,0), \) and \((0,0,1)\) oriented downward.

ii. \( \vec{F} = \langle yz^2, xz, y - z \rangle \) and \( S \) is the top-shaped surface given by \( z = \sqrt{x^2 + y^2} \) for \( 0 \leq z \leq 1 \) and \( z = 2 - \sqrt{x^2 + y^2} \) for \( 1 \leq z \leq 2 \) oriented outward.

**Problem 0.6.** (10 pts)

i. Let \( \vec{F} = \langle -2xz, y, z^2 - z \rangle \). Find a vector field of the form \( \vec{G} = \langle G_1, G_2, 0 \rangle \) such that \( \text{curl}(\vec{G}) = \vec{F} \).

ii. Let \( S \) be the surface \( z = 9 - x^2 - y^2, 5 \leq z \leq 9 \) oriented upward. Use part (i) to compute \( \int_S \vec{F} \cdot d\vec{A} \).

**Problem 0.7.** (10 pts) Suppose you are given two vectors \( \vec{v} \) and \( \vec{w} \) in \( \mathbb{R}^3 \) such that \( ||\vec{v}|| = 1, ||\vec{w}|| = 2, \) and the angle between them is 30 degrees.

i. Compute \( ||(\vec{v} + 3\vec{w}) \times \vec{w}|| \).

ii. Compute \( (\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) \).

iii. Is there a unit vector \( \vec{u} \) such that \( \vec{w} \cdot \vec{u} = 3? \) Explain.

**Problem 0.8.** (10 pts) Compute the flux of \( \vec{F} = \langle \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{-x}{\sqrt{x^2+y^2+z^2}}, z \rangle \) through the sphere of radius 1 centered at the origin oriented outward.

**Problem 0.9.** Compute the line integral \( \int_C \vec{F} \cdot d\vec{r} \) in the following cases:

i. \( \vec{F} = \langle yze^{xy}, xe^{xy}, e^{xy} \rangle \) and \( C \) is the line segment from \((1,0,0)\) to \((0,1,1)\).

ii. \( \vec{F} = \langle \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle \) and \( C \) is a circle of radius 2 centered at the origin, oriented counterclockwise.

**Problem 0.10.** (True/False, 2 pt each) Mark your answers True (T) or False (F). No explanation required!

i: \( \text{T \ [\ F] \} \). There is a function \( f(x,y) \) such that \( \int_0^1 \int_0^1 f(x,y) \, dx \, dy < 0. \)

ii: \( \text{T \ [\ F] \} \). \( \langle s - t, 3, 5t \rangle \) is the parametric equation of a plane.

iii: \( \text{T \ [\ F] \} \). There is a curve \( C \) in \( \mathbb{R}^2 \) such that \( \int_C \vec{F} \cdot d\vec{r} = 0 \) for every vector field \( \vec{F} \).

iv: \( \text{T \ [\ F] \} \). Given two non-zero vectors \( \vec{u}, \vec{v} \) in \( \mathbb{R}^3, \vec{u} \times \vec{v} = 0 \) implies that \( \vec{u} \) and \( \vec{v} \) are perpendicular.
v: T F. Suppose \( S \) is a surface and \( \vec{F} \) is a vector field that is everywhere perpendicular to the tangent plane of \( S \). It follows that \( \int_S \vec{F} \cdot d\vec{A} = 0 \).

vi: T F. If \( S \) is the upper half of a sphere of radius 3 then the divergence theorem can be used to evaluate the flux through \( S \).

vii: T F. The gradient points in the direction of the maximal rate of change of the function.

viii: T F. Fix a vector field \( \vec{F} \). The property: \( \int_C \vec{F} \cdot d\vec{r} = 0 \) around every closed curve \( C \) is equivalent to the property: \( \int_C \vec{F} \cdot d\vec{r} \) is a constant that depends only on the endpoints of \( C \) for any curve \( C \).

ix: T F. For a differentiable function \( f(x,y) \), \((\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, 1)\) is a normal vector for the tangent plane to \( f \).

x: T F. Let \( S \) be an oriented surface with oriented boundary \( C \). Fix vector field \( \vec{F} \). If \( \int_C \vec{F} \cdot d\vec{r} = 0 \) then \( \text{curl} \vec{F} = 0 \).

Problem 0.11.  
   i. Find a parametrization for a line through the points \((1, 3, 2)\) and \((-1, -3, 4)\).
   ii. Find a parametrization for a cylinder of radius 3 centered along the \( z \)-axis.
   iii. Do this line and cylinder intersect?

Problem 0.12. (5 pts) Write the volume of the region inside the cone \( z = \sqrt{x^2 + y^2} \) for \( 1 \leq z \leq 3 \) in spherical coordinates.

Problem 0.13. (15 pts) Write (but do not evaluate) the area of the region bounded by equations \( \frac{1}{2} \geq x \geq 0 \), \( y = 0 \), and \( x^2 + y^2 = 1 \) in
   i. Cartesian coordinates integrating \( y \) first.
   ii. Cartesian coordinates integrating \( x \) first.
   iii. Polar coordinates.

Problem 0.14. (10 pts) Convert:

   i. 
   \[
   \int_0^1 \int_{-1}^1 \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{(x^2 + y^2)^{3/2}} \, dy \, dx \, dz
   \]
   to cylindrical coordinates.

   ii. 
   \[
   \int_0^2 \int_{-x}^{2x} x + y \, dy \, dx
   \]
   to \( st \)-coordinates where \( x(s, t) = 2s + t \) and \( y(s, t) = s - t \).