Q1. (8 points) A box contains a $1 bill, a $5 bill, a $10 bill, and a $20 bill. A bill is selected at random, and is not replaced, then the second bill is selected at random. Draw a tree diagram and determine the sample space.

\[
\begin{array}{c|c}
1^{st} \text{ bill} & 2^{nd} \text{ bill} \\
\hline
& $5 \\
$1 & $10 \\
& $20 \\
$5 & $10 \\
& $20 \\
$10 & $5 \\
& $20 \\
$20 & $5 \\
& $10
\end{array}
\]

Sample space:

\[S = \{(\$1, \$5), (\$1, \$10), (\$1, \$20), (\$5, \$1), (\$5, \$10), (\$10, \$1), (\$10, \$5),
\] 

Q2. (2 points) If there are eight horses in a race, in how many different ways can they place first, second and third?

Permutation: order matters

\[8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336 \text{ ways}\]
Q3. (2 points) If 2 cards are drawn from a well-shuffled deck of 52 playing cards, what are the probabilities of getting

(a) two spades

\[ \frac{\binom{13}{2}}{\binom{52}{2}} = \frac{\left(\frac{13!}{(2!)(11)!}\right)}{\left(\frac{52!}{(2!)(50)!}\right)} = \frac{13}{221} \]

(note: \(13! = 13 \times 12 \times \ldots \times 1\))

(b) two aces

\[ \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{\left(\frac{4!}{(2!)(2!)}\right)}{\left(\frac{52!}{(50!)(2!)}\right)} \]

(a) a king and a queen

\[ \frac{\binom{4}{1} \times \binom{4}{1}}{\binom{52}{2}} = \frac{\left(\frac{4!}{3!1!}\right) \times \left(\frac{4!}{3!1!}\right)}{\left(\frac{52!}{(2!)(50)!}\right)} \]

(b) not a diamond

\[ \frac{\binom{39}{2}}{\binom{52}{2}} = \frac{\left(\frac{39!}{(2!)(37)!}\right)}{\left(\frac{52!}{(2!)(50)!}\right)} \]
Q4. (2 points) In the table shown below, 32 college students are classified according to their class standing and also according to their favorite pizza topping.

<table>
<thead>
<tr>
<th></th>
<th>Mushrooms ($M$)</th>
<th>Hamburger ($H$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freshman</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Sophomore</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>Junior</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

If one of these students is selected at random, $F$, $S$ and $J$ denote the three classes while $M$ and $H$ denote the two kinds of pizza toppings, find

(a) $P(M \cup J)$

$$P(M \cup J) = P(M) + P(J) - P(M \cap J) = \frac{12}{32} + \frac{13}{32} - \frac{5}{32} = \frac{20}{32} = \frac{5}{8}$$

(b) $P(M \cap S)$

$$P(M \cap S) = \frac{0}{32} = 0$$

(b) $P(M|F)$

$$P(M|F) = \frac{P(M \cap F)}{P(F)} = \frac{\frac{7}{32}}{\frac{10}{32}} = \frac{7}{10}$$

(d) $P(J|H)$

$$P(J|H) = \frac{P(J \cap H)}{P(H)} = \frac{\frac{8}{32}}{\frac{20}{32}} = \frac{8}{20} = \frac{2}{5}$$
Q5. (2 points) Determine whether the following can be probability distributions (defined in each case only for the given values of $x$). Please, explain your answer.

(a) $f(x) = P(X = x) = \frac{1}{6}$ for $x = 0, 1, 2, 3, 4, 5$

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
</tr>
</tbody>
</table>

$\sum P(x) = \frac{1}{6} \times (\frac{1}{6}) = 1$

Yes, this can be a probability distribution because $\sum P(x) = 1$ and $0 \leq P(X) \leq 1$ for all $X$.

(b) $f(x) = P(X = x) = \frac{x+1}{17}$ for $x = -1, 2, 3$

<table>
<thead>
<tr>
<th>$X$</th>
<th>-1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>$\frac{2}{17}$</td>
<td>$\frac{5}{17}$</td>
<td>$\frac{10}{17}$</td>
</tr>
</tbody>
</table>

$\sum P(x) = \frac{2}{17} + \frac{5}{17} + \frac{10}{17} = 1$

Yes, this can be a probability distribution because $\sum P(x) = 1$ and $0 \leq P(X) \leq 1$ for each $X$.

(c) $f(x) = P(X = x) = x$ for $x = 0.2, 0.3, 0.5$

<table>
<thead>
<tr>
<th>$X$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

$\sum P(x) = 0.2 + 0.3 + 0.5 = 1$

Yes, this is a probability distribution because $\sum P(x) = 1$ and $0 \leq P(X) \leq 1$ for each $X$.

(d) $f(x) = P(X = x) = \frac{x}{x+2}$ for $x = 0, 1, 2$

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

$\sum P(x) = \frac{1}{3} + \frac{1}{2} + 0 = \frac{5}{6}$

No, this cannot be a probability distribution because $\sum P(x) \neq 1$. 
Q6. (2 points) A bank vice president feels that each savings account customer has, on average, three credit cards. The following distribution represents the number of credit cards people own.

Number of cards   0   1   2   3   4
Probability       0.18 0.44 0.27 0.08 0.03

(a) Find the mean of this probability distribution. Is the vice president right?

\[ E = \sum (x \cdot p(x)) \]

\[ E = 0(0.18) + 1(0.44) + 2(0.27) + 3(0.08) + 4(0.03) \]

\[ E = 0 + 0.44 + 0.54 + 0.24 + 0.12 \]

\[ E = 1.34 \] credit cards

No, the bank vice president is not correct.

(b) Find the standard deviation of this probability distribution.

\[ \sigma^2 = \sum [(x - \mu)^2 \cdot p(x)] \]

\[ \sigma^2 = (0 - 1.34)^2 (0.18) + (1 - 1.34)^2 (0.44) + (2 - 1.34)^2 (0.27) \]

\[ + (3 - 1.34)^2 (0.08) + (4 - 1.34)^2 (0.03) \]

\[ \sigma^2 = 0.9244 \]

\[ \sigma = \sqrt{0.9244} = 0.9615 \]
Q7. (2 points) During the month of August, the daily number of persons visiting a certain tourist attraction is a random variable with \( \mu = 1200 \) and \( \sigma = 80 \).

(a) What does Chebyshev's theorem with \( k = 7 \) tell us about the number of persons who will visit the tourist attraction on an August day?

\[
P(\mu - k\sigma \leq X \leq \mu + k\sigma) \geq (1 - \frac{1}{k^2})
\]

\[
P(1200 - 7 \cdot 80 \leq X \leq 1200 + 7 \cdot 80) \geq (1 - \frac{1}{7^2})
\]

\[
P(1200 - 500 \leq X \leq 1200 + 500) \geq (1 - \frac{1}{49})
\]

\[
P(640 \leq X \leq 1760) \geq \frac{48}{49}
\]

\( \Rightarrow \) The probability that the number of persons will be equal to or greater than 640 and equal to or less than 1760 is at least \( \frac{48}{49} \).

(b) According to Chebyshev's theorem, with what probability can we assert that between 1000 and 1400 persons will visit the tourist attraction on an August day?

\[
P(\mu - k\sigma \leq X \leq \mu + k\sigma) \geq (1 - \frac{1}{k^2})
\]

\[
\mu - k\sigma = 1000
\]

\[
1200 - k(80) = 1000
\]

\[
k = \frac{1200 - 1000}{80} = 2.5
\]

\[
(1 - \frac{1}{k^2}) = (1 - \frac{1}{2.5^2}) = 1 - \frac{1}{6.25} = 0.84
\]

\( \Rightarrow \) \( P(1000 \leq X \leq 1400) \geq 0.84 \)

\( \Rightarrow \) The probability is at least 0.84. \( \checkmark \)
Q8. (2 points) Research shows that 70% of all women taking certain medication respond favorably. Using the Binomial distribution find the probabilities that among nine randomly selected women taking this medication

(a) at most two will respond favorably,

\[ P(X) = \frac{n!}{(n-x)!x!} \cdot p^x \cdot q^{n-x} \]

\[ p = 0.70 \quad q = 1 - 0.70 = 0.30 \quad n = 9 \quad x = 0, 1, 2 \]

Probability that 0 will respond favorably:

\[ \frac{9!}{(9!0!)} \cdot (0.70)^0 \cdot (0.30)^9 = (0.30)^9 \]

Probability that 1 will respond favorably:

\[ \frac{9!}{8!1!} \cdot (0.70)^1 \cdot (0.30)^8 = 9 \times 0.70 \times (0.30)^8 \]

Probability that 2 will respond favorably:

\[ \frac{9!}{7!2!} \cdot (0.70)^2 \cdot (0.30)^7 = 36 \times (0.70)^2 \times (0.30)^7 \]

\[ P(X = 0 \cup X = 1 \cup X = 2) = (0.30)^9 + \left[ 9 \times 0.70 \times (0.30)^8 \right] + \left[ 36 \times (0.70)^2 \times (0.30)^7 \right] \]

(b) at least one will respond favorably.

\[ P(\text{at least one will respond favorably}) = 1 - P(\text{0 will respond favorably}) \]

\[ = 1 - \left[ \frac{9!}{9!0!} \cdot (0.70)^0 \cdot (0.30)^9 \right] = 1 - (0.30)^9 \]
Q9. (2 points) The annual number of tornadoes in a certain state is a random variable with $\mu = 26.2$ and $\sigma = 4.1$. Approximating the distribution of this random variable with a normal distribution, find the probability that there will be at most 24 tornadoes in the given state in any given year. (Hint: note that the number of tornadoes is a discrete random variable, so you have to apply the continuity correction)

$$P(X \leq 24) = P(X < 24.5), \text{ using correction for continuity}$$

$$z = \frac{X - \mu}{\sigma} = \frac{24.5 - 26.2}{4.1} = -0.4146 \approx -0.41$$

$$P(z < -0.41) = 0.3409$$
Q10. (2 points) Find the area under the standard normal curve which lies

(a) between \( z = 0.48 \) and \( z = 1.98 \)

\[
\Phi(1.98) - \Phi(0.48) = 0.97701 - 0.6844 = 0.2927
\]

(b) between \( z = -1.36 \) and \( z = 1.06 \)

\[
\Phi(1.06) - \Phi(-1.36) = 0.8554 - 0.0809 = 0.7745
\]
Q11. (2 points) A park ranger wants to know the average size of trout taken from a certain lake. How large a sample of trout must be taken to be able to assert with a probability 0.96 that a sample mean will not be off by more than 0.5 inch. Assume that it is known from previous studies that $\sigma = 2.5$ inches.

\[ n = \left( \frac{Z_{\alpha/2} \cdot \sigma}{E} \right)^2 \]

\[ 1-\alpha = 0.96 \]
\[ \alpha = 0.04 \]
\[ \alpha/2 = 0.02 \]
\[ Z_{\alpha/2} = 2.055 \]

\[ n = \left( \frac{2.055 \times 2.5}{0.5} \right)^2 = 100 \text{ trout} \]

Q12. (2 points) A hospital wants to estimate the mean number of blood tests given to patients each day. Hospital records show that in a randomly selected sample of $n = 100$ days, the average number of blood tests was $\bar{x} = 83.5$, with standard deviation $s = 7.9$. Construct a 97% confidence interval for the mean number of blood tests given to patients each day.

\[ \bar{x} - Z_{\alpha/2} \left( \frac{s}{\sqrt{n}} \right) < \mu < \bar{x} + Z_{\alpha/2} \left( \frac{s}{\sqrt{n}} \right) \]

\[ n = 100 \quad \bar{x} = 83.5 \quad s = 7.9 \]

\[ 1-\alpha = 0.97 \]
\[ \alpha = 0.03 \]
\[ \alpha/2 = 0.015 \]
\[ Z_{\alpha/2} = 2.17 \]

\[ 83.5 - (2.17) \left( \frac{7.9}{\sqrt{100}} \right) < \mu < 83.5 + (2.17) \left( \frac{7.9}{\sqrt{100}} \right) \]
\[ 83.5 - 1.7143 < \mu < 83.5 + 1.7143 \]
\[ 81.7857 < \mu < 85.2143 \]

\[ 81.8 < \mu < 85.2 \quad \checkmark \]
Q13. (2 points) A service station advertises that customers will have to wait no more than 30 minutes for an oil change. A sample of 28 oil changes has a standard deviation of 5.2 minutes. Find 95% confidence interval of the population standard deviation of the times spent waiting for an oil change.

\[ n = 28 \quad df = n - 1 = 27 \quad s = 5.2 \quad s^2 = 27.04 \quad \frac{1 - \alpha}{2} = 0.025 \quad \alpha = 0.05 \]

Using chi-square distribution:

\[ \chi^2_{\text{right}} = 43.194 \]
\[ \chi^2_{\text{left}} = 14.573 \]

Q14. (2 points) A random sample of 12 graduates of a secretarial school type on average 72.6 words per minute with a standard deviation of 4.2 words per minute. Use the level of significance 0.05 to test an employer's claim that the school graduates average less than 75.0 words per minute.

\[ n = 12 \quad \bar{x} = 72.6 \quad s = 4.2 \quad \alpha = 0.05 \quad df = n - 1 = 11 \]

H₀: \( \mu = 75.0 \)

Hₐ: \( \mu < 75.0 \) (one tail)

\[ t = \frac{\bar{x} - \mu}{s / \sqrt{n}} = \frac{72.6 - 75.0}{(4.2 / \sqrt{12})} = -1.979 \]

Critical value = -1.796

Reject H₀ in favor of Hₐ. Evidence suggests that the employer's claim that school graduates average less than 75.0 words per minute is correct.
Q15. (2 points) For a certain year a study reports that the percentage of college students using credit cards was 83%. A college dean of student services feels that this is too high for her university, so she randomly selects 50 students and finds that 40 of them use credit cards. At $\alpha = 0.04$, is she correct about her university?

$$n = 50 \quad x = 40 \quad \hat{p} = \frac{40}{50} = 0.80 \quad \alpha = 0.04$$

$$p = 0.83 \quad q = 1 - p = 0.17$$

$$H_0: \ p = 0.83$$

$$H_A: \ p < 0.83$$

$$z = \frac{\hat{p} - p}{\sqrt{pq/n}} = \frac{0.80 - 0.83}{\sqrt{0.83 \times 0.17 \over 50}} = -0.560$$

Critical value = -1.75

$\Rightarrow$ Do not reject $H_0$. The college dean is not correct.

Q16. (2 points) A researcher claims that the standard deviation of the number of deaths annually from tornadoes in the United States is less than 35. If a sample of 11 randomly selected years had a standard deviation of 32, is the claim believable? Use $\alpha = 0.05$.

$$n = 11 \quad df = n - 1 = 10 \quad s = 32 \quad \sigma = 35 \quad \alpha = 0.05$$

$$H_0: \ \sigma = 35$$

$$H_A: \ \sigma < 35$$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(10)(32)^2}{(35)^2} = 8.359$$

Critical value = 3.940

$\Rightarrow$ Do not reject $H_0$. The claim is not believable.
Q17. (2 points) In a study of the relationship between family size and school performance in junior high school, 45 "only child" had an average GPA of 2.82 with a standard deviation of 0.34, and 60 firstborn children in two-child families had an average GPA of 2.96 with a standard deviation of 0.38.

(a) At the 0.12 level of significance, is the difference between these means significant?

\[ n_1 = 45 \quad \bar{x}_1 = 2.82 \quad s = 0.34 \]
\[ n_2 = 60 \quad \bar{x}_2 = 2.96 \quad s = 0.38 \]
\[ \alpha = 0.12 \quad \alpha/2 = 0.06 \]

\[ H_0 : \mu_1 = \mu_2 \quad \text{against} \quad H_A : \mu_1 \neq \mu_2 \]

\[ z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{2.82 - 2.96}{\sqrt{(0.34^2/45) + (0.38^2/60)}} = -1.98 \]

Critical values = ±1.555

Reject \( H_0 \). The difference between the means is significant.

(b) Determine the \( p \)-value.

\[ z = -1.98 \]

Critical values = ±1.98

\[ \Phi(-1.98) = 0.0239 \]

\[ 2 \times 0.0239 = 0.0478 = p \text{ value} \]

0.0478 > 0.12 → \( p \)-value < \( \alpha \), so reject \( H_0 \).
Q18. (2 points) To determine the effect of an advertised week-long sale on the numbers of personal computers sold, a chain of computer stores compared the number of PCs sold during the week prior to and during the week of the advertised sale, with the results given below. Use the 0.025 level to test whether the advertised sale was effective in increasing sales of PC’s.

<table>
<thead>
<tr>
<th>Store number</th>
<th>Week before $X_1$</th>
<th>Week of sale $X_2$</th>
<th>$O = X_1 - X_2$</th>
<th>$O^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24</td>
<td>29</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>54</td>
<td>62</td>
<td>-8</td>
<td>64</td>
</tr>
<tr>
<td>3</td>
<td>32</td>
<td>34</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>43</td>
<td>43</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>86</td>
<td>95</td>
<td>-9</td>
<td>81</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
<td>16</td>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>38</td>
<td>42</td>
<td>-4</td>
<td>16</td>
</tr>
</tbody>
</table>

$n = 7$  $df = n-1 = 6$

$$S_0 = \sqrt{\frac{n \sum O^2 - (\sum O)^2}{n(n-1)}} = \sqrt{\frac{7(199)-(0)^2}{7(6)}} = 4.276$$

$$\bar{O} = \frac{\sum O}{n} = \frac{-25}{7} = -3.57$$

$H_0: \mu_0 = 0$  $H_A: \mu_0 < 0$  $\alpha = 0.025$

$$t = \frac{\bar{O} - \mu_0}{S_0/\sqrt{n}} = \frac{(-3.57)}{4.276/\sqrt{7}} = -2.210$$

Critical value = -2.447

$\Rightarrow$ Do not reject $H_0$. The advertised sale was not effective in increasing sales of PC’s.
Q19. In a sample of 50 men, 44 said that they had less leisure time today that they had 1 year ago. In a sample of 50 women, 48 said that they had less leisure time today that they had 1 year ago. At \( \alpha = 0.10 \) is there a difference in the proportion?

\[
\begin{align*}
\hat{p}_1 &= \frac{x_1}{n_1} = \frac{44}{50} = 0.88 \\
\hat{p}_2 &= \frac{x_2}{n_2} = \frac{48}{50} = 0.96 \\
H_0: \ & \hat{p}_1 = \hat{p}_2 \\
H_A: \ & \hat{p}_1 \neq \hat{p}_2 \\
\alpha &= 0.10 \\
\frac{1}{2} &= 0.05
\end{align*}
\]

\[
\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{44 + 48}{100} = 0.92 \\
\bar{q} = 1 - \bar{p} = 1 - 0.92 = 0.08
\]

\[
Z = \frac{\hat{p}_1 - \hat{p}_2 - (\hat{p}_1 - \hat{p}_2)}{\sqrt{\bar{p} \cdot \bar{q} \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}} = \frac{(0.88 - 0.96)}{\sqrt{(0.92)(0.08)(\frac{1}{50} + \frac{1}{50})}} = -1.47
\]

Critical values = \( \pm 1.65 \)

\( \Rightarrow \) Do not reject \( H_0 \). There is not a difference in the proportion.
Q20. (2 points) The standard deviation of the ages of a sample of people who were playing roulette is 3.2 years. The standard deviation of the ages of a sample of people who were playing the slot machines is 6.8 years. If each sample contained 25 people, can it be concluded that the standard deviation of the ages are different? Use $\alpha = 0.10$.

\[
S_1 = 0.8 \\
\bar{x}_1 = 25 \\
df = 24
\]

\[
S_2 = 3.2 \\
\bar{x}_2 = 25 \\
df = 24
\]

\[
F = \frac{s_1^2}{s_2^2} = \frac{0.8^2}{3.2^2} = 0.5160
\]

$H_0: \sigma_1^2 = \sigma_2^2$

$H_A: \sigma_1^2 \neq \sigma_2^2$

$\alpha = 0.10$

$\alpha/2 = 0.05$

Critical Value = 1.98

$\Rightarrow$ Reject $H_0$. The standard deviations of the ages are different.