On H-Linked Graphs - Questions About Strong Connectivity

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One of the most natural and important properties in the study of graphs is **connectivity**.

**Definition**

A graph is connected if any two vertices are joined by a path.
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**Definition**

A graph is connected if any two vertices are joined by a path.

Connectivity itself has many ‘levels’. We say a graph $G$ of order at least $k$ is **$k$-connected** if the removal of any set of $k - 1$ vertices leaves a connected graph.
Equivalent Definition

**Definition**

$G$ is *k-connected* if for every two sets of $k$ vertices, say $X = x_1, \ldots, x_k$, and $Y = y_1, \ldots, y_k$, there are disjoint paths $P_1, \ldots, P_k$ such that $P_i$ joins a vertex of $X$ to a vertex of $Y$. 

```
  x_1 .
  x_2 .
  x_3 .
   .
   .
  x_{k-1} .
  x_k .
     .
     .
     .

  y_1 .
  y_2 .
  y_3 .
   .
   .
  y_{k-1} .
  y_k .
```
**Equivalent Definition**

**Definition**

If for every two sets of \( k \) vertices, say \( X = x_1, \ldots, x_k \), and \( Y = y_1, \ldots, y_k \), there are disjoint paths \( P_1, \ldots, P_k \) such that \( P_i \) joins a vertex of \( X \) to a vertex of \( Y \), then \( G \) is said to be \( k \)-connected.
Definition

**G is $k$-linked**

if for every sequence of $2k$ vertices, $x_1, \ldots, x_k, y_1, \ldots, y_k$, there are internally disjoint paths $P_1, \ldots, P_k$ such that $P_i$ joins $x_i$ and $y_i$. 

\[x_1 \quad \bullet \quad \bullet \quad y_1\\
x_2 \quad \bullet \quad \bullet \quad y_2\\
x_3 \quad \bullet \quad \bullet \quad y_3\\
\vdots \quad \vdots \quad \vdots \quad \vdots\\
x_{k-1} \quad \bullet \quad \bullet \quad y_{k-1}\\
x_k \quad \bullet \quad \bullet \quad y_k\]
**Definition**

$G$ is **$k$-linked**

if for every sequence of $2k$ vertices, $x_1, \ldots, x_k, y_1, \ldots, y_k$, there are internally disjoint paths $P_1, \ldots, P_k$ such that $P_i$ joins $x_i$ and $y_i$. 

![Diagram showing internally disjoint paths between pairs of vertices](image-url)
There are a lot of possibilities between $k$-connected and $k$-linked. Questions it seems no one has ever asked.

What if we partition $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $|X_i| = |Y_i|$, $i = 1, 2$. 
Question

There are a lot of possibilities between $k$-connected and $k$-linked. Questions it seems no one has ever asked.

What if we partition $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $|X_i| = |Y_i|$, $i = 1, 2$.

Then how connected must $G$ be so that there will exist disjoint paths $P_1, \ldots, P_{|X_1|}$ joining vertices of $X_1$ to vertices of $Y_1$ and disjoint paths $Q_1, \ldots, Q_{|X_2|}$ joining vertices of $X_2$ to $Y_2$?
Question

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Remark

Clearly, this is a concept that lies between connectivity and linkage. Also, many variations on this could be asked.

Can we distinguish these ideas with extra connectivity?

For now, let’s concentrate on $k$-linked graphs.
Alternate View

We can also view the $k$-linked problem as:

We are trying to find a subdivision of the graph $kK_2$ in a graph $G$, where we prescribe which vertices will play the role of the vertices in the $kK_2$. 
Alternate View

2t vertices selected

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Alternate View
If we can find a subdivision of a graph $H$ in a graph $G$ on any set of $|H|$ prescribed vertices of $G$, then we say that $G$ is $H$-linked.

The vertices of $G$ playing the roles of the vertices of $H$ are called the branch vertices of $G$.

Thus, $G$ is $k$-linked if and only if $G$ is $H = kK_2$-linked.
Jung and independently Larman and Mani - 1970

**Theorem**

*If $G$ is a $2k$-connected graph which contains a subdivision of $K_{3k}$, then $G$ is $k$–linked.*
Proof technique

$S(K_{3k})$

$x_1, x_2, ..., x_k, y_1, y_2, ..., y_k$
Proof technique

$S(K_{3k})$

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Proof technique

$S(K_{3k})$
Natural Question Arose

**Question**

Could enough connectivity alone imply a graph was $k$-linked?

**Remark**

Examples seemed to indicate something like $3k$-connected could be enough!
Example - not k-linked
Example - not $k$-linked

$K_{3k-1} - k K_2$

$K$ vertices missing

$k$ edges

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Question

What is the smallest function $f(k)$ so that every $f(k)$-connected graph is $k$-linked?

Question

Could it be that $f(k)$ is linear?
Bollobás and Thomason - 1996

Theorem

Let $H$ be a graph of order $p$ with $\delta(H) > p/2 + 4k - 2$. Then every $2k$-connected graph containing $H$ as a minor is $k$-linked.

Theorem

Every $22k$-connected graph is $k$-linked.
The following gives a connectivity bound that ensures a graph contains a subdivision of a given graph (in fact, many subdivisions).

**Theorem**

Let $H$ be a graph with $k$ vertices and $m$ edges. Then every $(22m + k)$-connected graph is $H$-linked.
Theorem

Every $2k$-connected graph with average degree at least $10k$ is $k$-linked.

Corollary

$f(k) \leq 10k.$
For $k = 1$, asking $G$ to be connected. Hence, $f(1) = 1$.
For $k = 2$, Jung showed $f(2) = 6$. In fact, he showed more.

**Theorem**

(a) Every maximally planar 4-connected graph is 2-linked.
(b) Every non-planar 4-connected graph is 2-linked.

Clearly, this implies that any 4-connected graph of order $n$ with at least $3n - 6$ edges is 2-linked.
Seymour and Thomassen - 1980 found the maximal non-2-linked graphs called \((S, T)\)-webs.
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Theorem

Let $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ be disjoint sets of vertices. A graph is maximal with respect to not having an $(S, T)$-linkage if and only if it is an $(S, T)$-web.
Chen, RG, Kawarabayshi, Pfender and Wei - 2005

**Theorem**

*Every 6-connected graph having \( K_9^- \) as a minor is 3-linked.*

Later, Thomas and Wollan - 2005 produced the optimal edge bound for ensuring a 6-connected graph is 3-linked.

**Theorem**

*Every 6-connected graph of order \( n \) and at least \( 5n - 14 \) edges is 3-linked.*
In view of the results on 2 and 3 linked graphs:

Problem

Does there exist a function $m(k)$ such that every $2k$-connected graph with at least $m(k)$ edges is $k$-linked?
Small \textit{k}

\textbf{Conjecture}

\textit{Every 8-connected graph is 3-linked.}
**Definition**

Let

\[ \sigma_2(G) = \min \{ \deg x + \deg y \mid xy \notin E(G) \}. \]
$R(n, k)$ denotes min pos integer $r$ such that every graph of order $n$ with $\sigma_2(G) \geq r$ is $k$-linked.

**Theorem**

If $k \geq 2$, then

$$R(n, k) = \begin{cases} 
2n - 3, & \text{if } n \leq 3k - 1, \\
\left\lfloor \frac{2(n+5k)}{3} \right\rfloor - 3, & \text{if } 3k \leq n \leq 4k - 2, \\
n + 2k - 3, & \text{if } n \geq 4k - 1.
\end{cases}$$
$D(n, k) = \min \text{ pos integer } d \text{ such that every graph of order } n \text{ and minimum degree } \delta(G) \geq d \text{ is } k\text{-linked.}$

Theorem

If $k \geq 2$, then

\[
D(n, k) = \begin{cases} 
 n - 1, & \text{if } n \leq 3k - 1, \\
\left\lfloor \frac{n+5k}{3} \right\rfloor - 1, & \text{if } 3k \leq n \leq 4k - 2, \\
\left\lceil \frac{n-3}{2} \right\rceil + k, & \text{if } n \geq 4k - 1.
\end{cases}
\]
Theorem

For every graph $H$, there is an integer $g(H)$ such that every graph $G$ with $\delta(G) \geq \max \{\Delta(H), 3\}$ and with girth at least $g(H)$ contains a subdivision of $H$.

Corollary

(a) Every $2k$-connected graph of sufficiently large girth is $k$-linked.
(b) Every $2\binom{n}{2}$-connected graph of sufficiently large girth contains a subdivision of $K_n$ with prescribed branch vertices.
Theorem

(a) For $k = 4$ or $5$, every $2k$-connected graph of girth 19 is $k$-linked.

(b) For $k \geq 6$, every $2k$-connected graph of girth at least 11 is $k$-linked.
Many sufficient conditions for a graph to be $H$-linked take a degree-based approach.

**Kostochka and Yu - 2005:**

**Theorem**

Let $H$ be a graph of order $k$ with $\delta(H) \geq 2$ and no loops. Every simple graph $G$ of order $n \geq 5n + 6$ with $\delta(G) \geq \left\lceil \frac{(n+k-1)}{2} \right\rceil$ is $H$-linked. This minimum degree condition is sharp.
Define $b(H)$:

- If $H$ is connected, $b(H)$ is defined to be the maximum size of an edge cut in $H$, unless $H$ contains no even cycles, then $b(H)$ is defined to be $|V(H)| - 1$.
- If $H$ has components $H_1, \ldots, H_t$, then $b(H) = u(H) + \sum_{i=1}^{t} b(H_i)$, where $u(H)$ denotes the number of components of $H$ that contain no even cycles.
H-Linked Graphs

- $k$ vertices in $X$
- $t$ vertices in $V(G) - X$
- $V(G) - X$
- $X$
- Max edge cut

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Theorem

Let $H$ be a connected multigraph, possibly containing loops. If $G$ is a simple graph of sufficiently large order $n$ with

$$\delta(G) \geq \left\lceil \frac{n + b(H) - 2}{2} \right\rceil,$$

then $G$ is $H$-linked. Furthermore, every injection $f : V(H) \rightarrow V(G)$ can be extended to an $H$-subdivision in which every edge-path has at most two intermediate vertices.
Kostochka and Yu - 2006

Theorem

Let $H$ be a loopless connected multigraph of order $k$ with $\delta(H) \geq 2$. If $G$ is a simple graph of order $n \geq 7.5k$ with

$$\delta(G) \geq \left\lceil \frac{n + b(H) - 2}{2} \right\rceil,$$

then $G$ is $H$-linked.
Theorem

Let $H$ be a multigraph of size $\ell$, possibly containing loops and let $k_1 = k_1(H) = \ell + u(H)$. If $G$ is a simple graph of order $n \geq 9.5(k_1 + 1)$ with

$$\delta(G) \geq \left\lceil \frac{n + b(H) - 2}{2} \right\rceil$$

then $G$ is $H$-linked. Furthermore, every injection $f : V(H) \to V(G)$ can be extended to an $H$-subdivision with at most $5k_1 + 2$ vertices.
Applications

(RG, Whalen, 2004)

**Theorem**

If $H$ is a multigraph and $G$ is a simple $(\max\{\alpha(H), \beta(H)\} + 1)$-connected graph of order $n > 11|E(H)| + 7(|H| - h_1(H))$ such that

$$\sigma_2(G) \geq n + |E(H)| - |H| + h_1(H) + h_0(H),$$

then $G$ is $H$-extendable.

$\alpha(H) =$ independence number
Applications

(RG, Whalen, 2004)

**Theorem**

If $H$ is a multigraph and $G$ is a simple $(\max\{\alpha(H), \beta(H)\} + 1)$-connected graph of order $n > 11|E(H)| + 7(|H| - h_1(H))$ such that

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Applications

(RG, Whalen, 2004)

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$h_0 =$ number of vertices of deg 0
Applications

(Theorem, RG, Whalen, 2004)

If $H$ is a multigraph and $G$ is a simple $(\max\{\alpha(H), \beta(H)\} + 1)$-connected graph of order $n > 11|E(H)| + 7(|H| - h_1(H))$ such that

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then $G$ is $H$-extendable.

$h_1 =$ number of vertices of deg 1
Applications

(RG, Whalen, 2004)

Theorem

If $H$ is a multigraph and $G$ is a simple $(\max\{\alpha(H), \beta(H)\} + 1)$-connected graph of order $n > 11|E(H)| + 7(|H| - h_1(H))$ such that

$$\sigma_2(G) \geq n + |E(H)| - |H| + h_1(H) + h_0(H),$$

then $G$ is $H$-extendable.

$H$-extendable = spanning subdivision of $H$ in $G$
Definition

By an independent edge set we mean a set $E$ of edges whose end vertices have degree one in the graph induced by $E$.

**Note:** Since we consider a loop as adding two to the degree of a vertex, loops are not considered independent edges. Similarly, since the graph induced by a single vertex with a loop has an edge, we do not consider such vertices as independent.
Corollary

(Ore, 1960) Let $G$ be a graph on $n \geq 3 (> 18)$ vertices with $\sigma_2(G) \geq n$, then $G$ is hamiltonian.

Proof. Let $H$ be a single vertex with a loop. Then $|E(H)| = |V(H)| = 1$ and $h_1 = h_0 = 0 = \alpha = \beta$. Clearly, the degree condition implies the existence of a cycle.
Corollary

*(Ore)* Let $G$ be a connected graph of order $n \geq 3(\geq 12)$ and $\sigma_2(G) \geq n + 1$, then $G$ is hamiltonian connected.

Proof: Let $H = K_2$. Then $|E(H)| = 1$, $|H| = 2$ and $h_1 = 2$ and $h_0 = 0$. Also $\alpha = 1 = \beta$.

Thus, using the theorem we get $\sigma_2 \geq n + 1 - 2 + 2 = n + 1$ and that $n > 11(1) + 7(2 - 2) = 11$. As $G$ is connected, there exists a path between any two vertices, and by the theorem, it extends to a hamiltonian path.
### Definition

A graph $G$ is *$k$-ordered* if given any sequence of $k$ vertices, say $x_1, x_2, \ldots, x_k$, there is a cycle in $G$ where these vertices occur in this order (in one direction or the other).

### Definition

$G$ is *$k$-ordered hamiltonian* if it is $k$-ordered and if the cycle containing these $k$ vertices spans $V(G)$. 
Corollary

(Kierstead, Sárközy and Selkow, 1999)

If $G$ is a graph of order $n \geq 11k - 3$ ($n > 18k$) with 
$\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$, then $G$ is $k$-ordered hamiltonian.

Proof. Use $H = C_k$, then $b(H) = k$ (or $k - 1$) and in any case the bound on $\delta$ for the $H$-linked result provides the above bound exactly. Thus, $G$ is $k$-ordered. Further, this bound exceeds that for the extension theorem $(n/2)$, hence $G$ is $k$-ordered hamiltonian.
Corollary

(Brandt, et. al, 1997) Let $G$ be a graph of order $n$ with $\sigma_2 \geq n$. Then for all $1 \leq k \leq n/4(\leq n/18)$, we have that $G$ contains a 2-factor with exactly $k$ cycles.

Proof: Let $H$ be $k$ vertices, each with a loop. Then $|E(H)| = |V(H)| = k$ and $h_1 = \alpha = \beta = 0$. Justesen’s Theorem implies $k$ vertex disjoint cycles exist under this degree condition. The Extension Theorem then implies they extend to form a 2-factor.
Another Generalization - Immersions

- Suppose we weaken the question.
Another Generalization - Immersions

- Suppose we weaken the question.

- Suppose we now wish to allow the paths in $G$ to be **edge disjoint** but **not necessarily vertex disjoint**.
Another Generalization - Immersions

- Suppose we weaken the question.
- Suppose we now wish to allow the paths in $G$ to be edge disjoint but not necessarily vertex disjoint.
- Call such a graph an $H$-immersion in $G$. 
Immersions

\[ G \]

\[ f(x) \]

\[ H \]

\[ x_1 \]
\[ x_2 \]
\[ x_3 \]
\[ x_4 \]
\[ x_5 \]

\[ f(x_1) \]
\[ f(x_2) \]
\[ f(x_3) \]
\[ f(x_4) \]
\[ f(x_5) \]

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Immersions

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Immersions
Definition For an immersion $I$ of $H$ in a graph $G$, let $S = f(V(H))$. For a vertex $v \in G - S$, define the **vertex repetition number**, $r(v, I)$, to be one less than the number of paths of $I$ containing $v$ (and zero if $v$ is not used in forming $I$).

Definition Define the **vertex repetition number** $r(I)$ to be the sum of the vertex repetition numbers of vertices of $G - S$. 
Immersion Result


**Theorem**

Let $H$ be a multigraph of order $k$ and $G$ a simple graph of sufficiently large order $n$. If $\lambda$ is an integer such that $0 \leq \lambda \leq b(H) - k + 1$, and

$$\delta(G) \geq \frac{n + b(H) - \lambda - 2}{2},$$

then any injective map $f : V(H) \rightarrow V(G)$ can be extended to an $H$-immersion, $I$, with $r(I) \leq \lambda$. 
A graph $G$ is weakly $k$-linked if given any two sets of $k$ distinct vertices, $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$ we can find $k$ pairwise edge-disjoint paths $P_1, P_2, \ldots, P_k$ such that $P_i$ joins $s_i$ to $t_i$, $i = 1, 2, \ldots, k$. 

![Diagram illustrating a weakly k-linked graph]
### Definition

A graph $G$ is *weakly $k$-linked* if given any two sets of $k$ distinct vertices, $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$, we can find $k$ pairwise edge-disjoint paths $P_1, P_2, \ldots, P_k$ such that $P_i$ joins $s_i$ to $t_i$, $i = 1, 2, \ldots, k$.
Conjecture - Thomassen 1980

**Conjecture**

There exists a function $g(k)$ such that every $g(k)$-edge connected multigraph is weakly $k$-linked.

He also showed that $k$-edge connected was not enough for a multigraph to be weakly $k$-linked.
Conjecture

If $k$ is odd, then $g(k) = k$ and if $k$ is even, then $g(k) = k + 1$.  

Thomassen - 1980

**Theorem**

\[
g(k) \leq k + 1 \quad \text{if } k \text{ is odd, and} \\
g(k) \leq k + 2 \quad \text{if } k \text{ is even.}
\]