Linear Bounds for Graph Coloring: A Unifying Theory
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We will discuss how linear bounds in graph coloring lead to many new and interesting results. To that end, we say a family $\mathcal{F}$ of graphs embedded in surfaces is hyperbolic if for every $G \in \mathcal{F}$ and for every closed curve $\gamma$ that intersects $G$ only in vertices and bounds a planar disk $\Delta$, we have that $|V(G) \cap \Delta| \leq O(|V(G) \cap \gamma|)$. This says that the number of vertices inside such a planar disk is linear in the number of vertices on the boundary of that disk. Similarly we say that a family $\mathcal{F}$ is strongly hyperbolic if the same holds for every annulus $\Delta$. Being strongly hyperbolic has a number of interesting consequences. Foremost is the fact that the number of vertices of a graph in a strongly hyperbolic family is linear in the Euler genus of the surface. This gives rise to a linear-time algorithm for testing whether a graph on a fixed surface contains a member of $\mathcal{F}$.

The concept of hyperbolicity unifies and simplifies many known results about coloring graphs on surfaces while resolving a number of open conjectures. Some of these follow from our recent result that 6-list-critical graphs are strongly hyperbolic. In particular, the theory of strongly hyperbolic families then implies that the number of 6-list-critical graphs on a fixed surface is finite, resolving a conjecture of Thomassen. It also follows that locally planar graphs with distant precolored vertices are 5-choosable. For the plane, this was conjectured by Albertson and recently resolved by Dvorak, Lidicky, Mohar and Postle. Furthermore, we can prove that locally planar graphs drawn in a surface with crossings far apart are 5-choosable which generalizes a result of Dvorak, Lidicky and Mohar for the plane. As for a different example, Dvorak and Karawabayashi have shown that 4-list-critical graphs of girth at least five are hyperbolic. Applying our theory, we obtain the new result that locally planar graphs of girth five are 3-choosable. This is joint work with Robin Thomas.