Chapter 3 - Review of Preliminary Set Theory

Our notion of probability will be derived entirely from counting the size of sets and the operations that one can perform on sets. For us a set will be a collection of objects called elements. Sets can be described by a simple listing of each member of the set or by describing the set in more general terms. The set with no elements is called the empty set and is denoted by \( \emptyset \).

\[
\{1, 2, 3, 4, 5\} \\
\{\text{red, white, blue}\} \\
a \text{deck of playing cards with jokers} \\
\{1, 4, 9, 16, 25, 36...\}
\]

All of the above are perfectly good examples of sets. These are each very different from one another. The first three sets are finite while the last is infinite. The third set is defined by description but it is perfectly clear what its members are. The last set could have been described as the squares of all positive integers. The description of a set is by no means unique. If \( A \) and \( U \) are sets and every element of \( U \) is an element of \( A \) then \( U \) is a subset of \( A \). The sample space is the universal set in which a given problem is set. It contains all possible elements for the given situation. The sample space is generally denoted \( S \).

Let \( S = \{1, 2, 3, 4, 5\} \) and \( U = \{1, 3, 4\} \). Then \( U \) is a subset of \( S \). We denote this by \( U \subseteq S \). The number 3 is an element of \( S \). We denote this by \( 3 \in S \). We can negate any shorthand by a slash through the symbol. The number 8 is not a member of \( S \), which is abbreviated by \( 8 \notin S \).

Now that we have these sets we’d like to be able to manipulate them. To that end we have the following set operations.

For any sets \( A \) and \( B \) in a sample space \( S \):

\( A \cup B \), **A union B**, is the set of all elements in \( S \) that are in set \( A \) or in set \( B \) or in both.

\( A \cap B \), **A intersect B**, is the set of all elements of \( S \) that are in set \( A \) and in set \( B \).

\( A' \), **A complement**, is the set of all elements of \( S \) that are not in \( A \).

Consider \( S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), \( P = \{1, 7, 8, 9, 10\} \) and \( Q = \{1, 2, 3, 4, 5\} \). Then, \( P \cap Q = \{1\} \) because 1 is the only element which both sets contain, while \( P \cup Q = \{1, 2, 3, 4, 5, 7, 8, 9, 10\} \). The complement of \( P \) is the set \( P' = \{2, 3, 4, 5, 6\} \). Naturally, we’re not limited to one operation at a time and we can mix and match to our heart’s content. For example, \((P \cup U)' = \{6\} \).

It may seem most natural to perform set operations on numbers but the power will come when we have sets with more descriptive meaning to games. Consider the sample space \( S \)
to be a standard deck of playing cards. Let \( A \) be the set of aces, \( H \) be the set of hearts and \( C \) be the set of clubs. The set \( A \cap H \) is the ace of hearts and the set

\[
A \cap (H \cup C) = \{\text{ace of hearts, ace of clubs}\}.
\]

Sets \( H \) and \( C \) have no elements in common, thus \( H \cap C = \emptyset \).

It may also be very useful when working with set operations to employ Venn diagrams. Venn Diagrams can make some set equalities obvious, when the formulas themselves may not be as clear. Case in point, consider the sets \( (A \cup B)' \), \( (B \cap A)' \) and \( (A' \cap B') \).

\[
\begin{align*}
\text{B intersect A'}
\end{align*}
\]

The **order** of a set \( A \) is the number of elements in the set, denoted \(|A|\). Let \( S \) be the set of all positive integers that are even and less than 15. Clearly, \(|S| = 7\). If \( C \) is the set of playing cards with no jokers then \(|C| = 52\). We get an interesting relationship between the orders of \( A \cup B \) and \( A \cap B \). When counting the number of elements in \( A \) and \( B \), you certainly count each element in either \( A \) or \( B \). An element that is in both sets \( A \) and \( B \) will be counted twice with regard to \(|A|\) and \(|B|\) and we only wish to count it once with regard to \(|A \cup B|\). Thus we get the following formula for any two sets \( A \) and \( B \):

\[
|A \cup B| = |A| + |B| - |A \cap B|.
\]

**Example 1** On a test 12 students made an "A" or answered the bonus question correctly. If 7 students made an "A" while 10 students answered the bonus correctly, how many students made an "A" and answered the bonus correctly? Using the above result, let \( A \) be the set of students who make an "A" and \( B \) be the set of students who answered the bonus correctly. Then, the answer to the question will be the order of \( A \cap B \).
Applying the equation earns us $12 = 7 + 10 - |A \cap B|$. Accordingly our answer is 5.

**Example 2** Of 200 movie goers surveyed 123 always bought candy or popcorn. Of those people, 45 always bought only candy while 51 always bought popcorn. How many people always buy both popcorn and candy?

Using Venn diagrams it’s easy to compute the intersection to be:

$$123 - (45 + 51) = 27.$$

**Exercises 0.1**

1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ be the sample space.

   Let $P = \{2, 4, 6, 8, 10, 12, 14\}$ and let $Q = \{3, 6, 9, 12, 15\}$.

   Write out the following subsets:

   - $P \cup Q$, $P \cap Q$, $Q \cap P'$, $P \cap Q'$
   - $P' \cup S$, $P \cup S$, $Q \cap S$, $P' \cap S$.

2. Illustrate the following by shading in a Venn diagram:

   $A' \cap B$, $A' \cap B'$, and $P' \cup Q$.

   What is another way of describing this last set?

3. Let $A$ be the set of odd positive integers less than 100. Let $B$ be the set of positive integers less than 100 which are divisible by 3. Compute:

   $$|A \cap B| \text{ and } |A \cup B|.$$

4. Let $H$ be the set of students who take History 102 and $B$ the set of students who take Biology 112. Assume that $|H| = 40$ and $|B| = 45$. What is $|H \cup B|$ if

   (a) History 102 and Biology 112 meet at the same time?

   (b) History 102 and Biology 112 do not meet at the same time and there are 15 students who take both?
5. Let $S$ be a sample space and $A$ and $B$ be any two subsets of $S$.

$$A' \cap S = \_\_\_.$$  

$$B \cup S = \_\_\_.$$  

$$(A' \cap B)' \cap S = \_\_\_. $$

6. In a survey, 100 people drink Coke or Pepsi, 70 drink Coke while 45 drink Pepsi. How many people drink both Coke & Pepsi?

7. The Nielson ratings claim 232,679 people watch The Brady Bunch and Andy Griffith every day while 532,568 watch either program. If 392,101 people watch Andy, how many people watch The Brady Bunch?

8. In a survey at Emory of 200 students who say they drink beer or wine, 70 drink only beer while 45 drink only wine. How many people drink both beer and wine? Explain the difference between this problem and problem #6.
Chapter 4 Start Counting

4.1 Counting

The key to successful computation of probabilities is the ability to count. Unfortunately, this is harder than it sounds. Let’s consider a social club that consists of three individuals, Mark, Karen, and Pat. Every club needs a president and a vice-president, so the question becomes how many different combinations of president and vice-president can we choose for this small club. For this small a club it is very easy to write down all the possible ways:

<table>
<thead>
<tr>
<th>president</th>
<th>vice-president</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mark</td>
<td>Karen</td>
</tr>
<tr>
<td>Mark</td>
<td>Pat</td>
</tr>
<tr>
<td>Karen</td>
<td>Mark</td>
</tr>
<tr>
<td>Karen</td>
<td>Pat</td>
</tr>
<tr>
<td>Pat</td>
<td>Mark</td>
</tr>
<tr>
<td>Pat</td>
<td>Karen</td>
</tr>
</tbody>
</table>

Clearly, the answer to our question is 6. But, what if our club had 20 members, or 5000? We certainly would not want to attempt to write down all combinations in these cases! We wouldn’t even be certain that we had found all combinations. We would like a better counting technique than primitive brute force. Upon closer analysis we see that we had three choices for whom to pick as president. For each choice of president we are left with two choices as vice-president. It’s no coincidence that $3 \times 2 = 6$. If our club consisted of 20 people we would have $20 \times 19 = 380$ ways of choosing the president and vice-president combination. This is the fundamental idea behind the:

**Multiplication Principle:**

If task 1 can be performed in $n_1$ ways and task 2 can be performed in $n_2$ ways regardless of how task 1 was performed then the total number of ways of performing task 1 and then task 2 is $n_1 \times n_2$.

Now if we wish to know how many license plates exist with three letters followed by three digits we can use the multiplication principle to find that there are $26^3 \times 10^3 = 17,576,000$.

The careful reader is now asking what the “regardless of how” phrase of the multiplication principle implies. This principle holds only if regardless of how we act, there are always the same number of choices. Consider the 20 person club made up of 8 men and 12 women. Further let’s say that in today’s politically correct climate we must pick a president and a vice-president of opposite gender. Now we still have 20 choices for the president but the sticky point arises when we go to choose a vice-president. If we picked a male president then
we have 12 choices for a female vice-president. Conversely if we picked a female president then we have 8 choices for a male vice-president. In this case the number of ways task 2 can be performed changes based on the choice for task 1. In order to use the multiplication principle here we must break the problem in three tasks. Task 1 is to pick a male to serve in office, task 2 is to pick a female to serve in office and task 3 is to determine which gender will be president (this automatically selects the other gender to serve as vice-president). Hence, we can select our president and vice-president of opposite gender in \( 8 \times 12 \times 2 = 192 \) ways.

**Example 3** You and a date are going to order pizza for dinner. Toppings offered by the restaurant are mushrooms, sausage, pepperoni, anchovies, onions and peppers. How many different combinations of toppings can you order for your pizza?

**Solution:** There are 6 potential toppings and for each your only decision is do we have it or not. This breaks down to 6 tasks to perform with each task having a choice of two ways (to have the topping or not). Thus, there exists \( 2^6 = 64 \) different ways to choose toppings for your pizza. This runs the gamut of a plain pizza to all the way. We currently cannot determine how many 2-item pizzas we could order.

**Example 4** How many different batting orders exist for the nine positions in baseball (do not assume that the pitcher will bat 9\textsuperscript{th})? We have nine tasks to perform. We pick a position to bat first and then a position to bat second, etc. Clearly there are \( 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880 \) ways to do this.

Each different ordering of these nine positions is called a **permutation**. The product of all the integers from 1 to \( k \) is denoted as **\( k \)-factorial**. The total number of permutations of \( k \) objects is \( k \) factorial. Rather than write the product of \( k \) integers every time we need a factorial, we use an exclamation mark to denote a factorial.

\[
k! = k \times (k - 1) \times (k - 2) \times \ldots \times (3) \times (2) \times (1) \text{ for every positive integer } k.
\]

As a matter of convenience we define \( 0! = 1 \).

**Example 5** Compute the number of different permutations and subsets of the characters \{a, b, c, d\}.

*There are four characters so there will be \( 4! = 24 \) different permutations.*
\[
\begin{array}{ccc}
abcd & abdc & acdb \\
acbd & adbc & adb \\
bcad & badc & bcad \\
bdca & bdac & bdca \\
cabd & cadb & cbad \\
bcda & cdab & cdba \\
dabc & dacb & dbac \\
dcba & dcab & dcba \\
\end{array}
\]

Just as in the pizza problem there are \(2^4 = 16\) different subsets of these characters.

\[
\emptyset \\
\{a\}, \{b\}, \{c\}, \{d\} \\
\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\} \\
\{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\} \\
\{a, b, c, d\}
\]

The difference between permutations and subsets is clear. For permutations, the elements and their order is important while when considering subsets, only the elements themselves are important. These two differences will be critical when computing probabilities.

**Exercises 1.1**

1. How many different passwords exist using four lowercase letters followed by two digits (0 – 9)?

2. How many different passwords of 7 characters exist where each character may be a lowercase letter of the alphabet or a digit? How many if we allow upper and lowercase letters?

3. How many different license plates consisting of three distinct letters followed by three distinct numbers exist?

4. A restaurant offers a daily lunch special. The special consists of a salad with a choice of 10 dressings, 5 different sandwiches, 3 different desserts and a drink choice consisting of water, tea, Coke or Diet Coke. How many different lunches exist?

5. This same restaurant offers pizza with eight different toppings available. You can order a large, medium, or small pizza with any combination of toppings and any number of toppings. You may not order any topping twice. How many pizzas can be ordered?
6. How many different ways can you rearrange the letters in the word "riot"? Find them all.

7. How many different ways can you rearrange the letters in the word "stygian"?

8. An ice cream store allows you to order your choice of one scoop of their 51 different flavors, choice of waffle cone or sugar cone, and a choice of 10 different toppings. How many different ways can you order an ice cream?

9. How many subsets of the set \{a, b, c\} exist? Write them all out. How many permutations of the set \{a, b, c\} exist? Write them all out.

10. How many subsets of the set \{a, b, c, d, e, f, g, h, i\} exist? How many permutations of the set \{a, b, c, d, e\} exist?

11. You have a blue die, a red die and a purple die. How many different ways can you roll them?

12. Two men and two women go to lunch together in one car with two people in the front and two in the back. How many different ways can they sit in the car (assume it does not matter who drives)? How many different ways can they sit if members of the opposite sex must sit together either in front or in back?

13. You flip a fair coin four times. How many different ways can you do this? Write them all out.

4.2 Combinations

Let’s consider our pizza problem again. We still have six toppings to choose from. However, this time we wish to count how many two item pizzas exist. Our initial approach is to say that we have two tasks, pick topping number one and then pick a different topping number two. This would be done $6 \times 5 = 30$ ways. However, closer analysis is needed here.

<table>
<thead>
<tr>
<th>topping #1</th>
<th>topping #2</th>
</tr>
</thead>
<tbody>
<tr>
<td>onions</td>
<td>peppers</td>
</tr>
<tr>
<td>peppers</td>
<td>onions</td>
</tr>
</tbody>
</table>

These two cases are considered different in the 30 ways because we are implying that topping number 1 is different from topping number 2. As any experienced pizza connoisseur will tell you that is not the case! One approach we can take here is to get rid of the repetition. Each
pair of toppings will give rise to two orders as the example above does. So the true number of two item pizzas is actually \((6 \times 5)/2 = 15\).

The other approach is to develop a method for picking objects as a set without implying some order or difference to them.

How many ways can we pick \(k\) objects from \(n\) objects as a set with no member having more importance than any other member? Using the above strategy we need to pick object number’s one, two, \ldots, \(k\). We can do this in

\[ n \times (n - 1) \times (n - 2) \times \ldots \times (n - k + 1) \text{ ways.} \]

However we will still get repetitions like we did in the pizza problem. Each set of \(k\) objects occurs in \(k!\) different ordered sets. So the total ways is:

\[ \frac{n \times (n - 1) \times (n - 2) \times \ldots \times (n - k + 1)}{k!} \]

which, with a little manipulation can be written as

\[ \frac{n!}{(k! \times (n - k)!)} \text{.} \]

**Fact 6** The number of ways of choosing \(k\) distinct objects as a set from \(n\) distinct objects is given by the **binomial coefficient**

\[ \binom{n}{k} = \frac{n!}{k!(n - k)!} \text{.} \]

From our twenty person club how many ways can we pick a three person committee with all three individuals of equal rank and power? Here we want the binomial coefficient twenty choose three:

\[ \binom{20}{3} = \frac{20!}{3! \times 17!} = \frac{20 \times 19 \times 17}{3 \times 2 \times 1} = 1140 \text{.} \]

Let’s finally consider a game and take a brief glimpse at Poker. How many different poker hands exist and how many of those are four-of-a-kind? In an ordinary deck of 52 playing cards (no jokers unless specifically stated) a poker hand is just a subset of 5 cards. Hence there are

\[ \binom{52}{5} = \frac{52!}{5! \times 47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960 \text{.} \]
different poker hands.

How many are four of a kind? A four of a kind hand consists of one number (2, 3, 4, …, 10, J, Q, K, A) represented by all four of its suits and one other card. So we must pick the rank (number) to get four of and then get four of that rank and then we must pick the fifth card. This can be done in exactly

$$13 \times \binom{4}{4} \times 48 = 624$$

ways.

**Exercises 4.2**

1. Out of a 40 person club, how many different ways can you choose a president, vice-president, and secretary? How many different ways can you pick a three person committee?

2. My cd collection consists of 5 Cure cd’s, 3 R.E.M. cd’s, 2 Nirvana, 2 B-52’s and 1 Frank Sinatra cd. I’m planning a trip and randomly take 4 cd’s. How many different ways can I do this? How many different ways if the Sinatra cd must be one of the four?

3. How many different ways can you rearrange the letters in the word ”book”?

4. How many different ways can you rearrange the letters in the word ”level”?

5. An ice cream store allows you to order your choice of two scoops (no doubles of one flavor) of their 51 different flavors, choice of waffle or sugar cone, and choice of one of their 10 different toppings. How many different ways can you order an ice cream?

6. An ice cream store allows you to order your choice of two scoops (doubles allowed) of their 51 different flavors, choice of waffle or sugar cone, and choice of one of their 10 different toppings. How many different ways can you order an ice cream?

7. You have three copies of Dickens’ *David Copperfield* to give as Christmas presents at the local orphanage. There are currently 10 orphans present. How many different ways can you pick three different children to receive a book?

8. Let’s say you also have three copies of Camus’ *The Stranger* to distribute. How many different ways can you distribute the six books if no orphan may receive a copy of each book? How many different ways if an orphan may receive a copy of each book?

9. There are 12 other people at the playground waiting to play basketball. You need to pick four of them to make up the rest of your team to challenge the current winners. How many different ways can you do this? How many different ways can you do this and determine who plays the two guard positions, two forward positions, and center?
Now assume that Michael Jordan is at the playground (and its a given you will pick him to be one of your guards), now how many ways can you pick your team?

10. A group of people consists of four men and three women. They go to the theater and see a row of seven empty seats.
   (a) How many different ways can they sit down?
   (b) How many different ways can they sit down if members of the same sex cannot sit beside one another?
   (c) How many different ways can they sit down if all the men are to sit together and all of the women are to sit together?

4.3 Probability

Consider a multiple choice question with four possible replies. If an individual does not know the answer and is forced to randomly guess the answer what is the probability of picking the correct answer? With one correct answer and four options the probability is one out of four or 25%. It is this concept that is the heart of probability.

**Definition 7** For a given experiment and an event A (a subset of the outcomes of this experiment) the probability of A occurring is given by \( P(A) = \frac{S}{N} \) where \( S \) is the number of outcomes of the experiment that satisfy \( A \) and \( N \) is the total number of outcomes of the experiment and each outcome is equally likely to occur.

A class of 20 people is composed of 12 women and 8 men. Two students are picked at random to be exempt from the final exam. The probability that a student of each gender is picked will be the number of ways to pick a student of each gender over the number of ways to pick two students from the class. The probability will be

\[
\frac{12 \times 8}{\binom{20}{2}} = \frac{96}{190} = .505
\]

In the previous section we computed the number of different poker hands and how many of those were four of a kinds. There is now no work involved in computing the probability of a four of a kind. It is simply \( P(\text{four of a kind}) = \frac{13 \times \binom{4}{1} \times 48}{\binom{13}{5}} \).

We will concentrate on the other poker hands in the next section.
A careful review of the definition of probability yields the warning that all outcomes of the experiment must be equally likely. This must be taken into consideration when computing probabilities! Judge the experiment of rolling a pair of fair dice and the probability of rolling a sum of 7. The reckless novice will take $S = 1$ and $N = 11$ with the logic that there are only eleven possible sums on two six sided dice ($2, 3, \ldots, 10, 11, 12$) and that 7 is one of them. This violates our definition that all outcomes must be equally likely! There is only one way to attain the sum of 12 yet many more ways to sum up to 7.

So, we must take a different approach. The correct method is to consider all ordered pairs of the number $1 - 6$.

$$(1,1) \ (1,2) \ (1,3) \ (1,4) \ (1,5) \ (1,6)$$
$$(2,1) \ (2,2) \ (2,3) \ (2,4) \ (2,5) \ (2,6)$$
$$(3,1) \ (3,2) \ (3,3) \ (3,4) \ (3,5) \ (3,6)$$
$$(4,1) \ (4,2) \ (4,3) \ (4,4) \ (4,5) \ (4,6)$$
$$(5,1) \ (5,2) \ (5,3) \ (5,4) \ (5,5) \ (5,6)$$
$$(6,1) \ (6,2) \ (6,3) \ (6,4) \ (6,5) \ (6,6)$$

This simulates the rolling of two dice and each of these $6 \times 6 = 36$ ways is equally likely. Six of these entries sum up to seven. Hence, the probability of rolling the sum of seven is $6/36 = 0.167$. Also easy to compute is the probability of rolling a sum of 9. Just check to see how many entries sum up to 9 and divide by 36. We get $4/36 = 0.112$. The moral is that probability is nothing more than counting the correct number of ways, but we must be careful not to violate our "all events are equally likely" definition.

**Exercises 4.3**

1. You roll one die. What is the probability you roll a number larger than a 2?

2. You roll a pair of dice. What is the probability that you roll a sum of 6? A sum of 11?

3. You roll a pair of dice. What is the probability that the sum is a prime number? What is the probability the sum is 10 or more? What is the probability of doubles?

4. You flip a fair coin twice. Write out the sample space. What is the probability of exactly one head?

5. You flip a fair coin three times. Write out the sample space. Find the probability of exactly two heads. Find the probability of two or more tails.

6. In a certain class there are 15 females and 12 males. I randomly pick five students for a quiz. What is the probability I pick exactly three females?
7. My cd collection consists of 5 Cure cd’s, 3 R.E.M. cd’s, 2 Nirvana, 2 B-52’s and 1 Frank Sinatra cd. I’m planning a trip and randomly take 5 cd’s. What is the probability that I take exactly 2 Cure cd’s?

A deck of cards consists of 52 cards of four suits, hearts, clubs, spades, diamonds. Each suit consists of one each of 2, 3, . . . , 9, 10, J, Q, K, A.

8. You pick three cards from the deck. What is the probability they are all spades? What is the probability they are all the same suit?

9. You pick two cards from the deck. What is the probability they are both kings? What is the probability they are the same number?

10. You pick four cards from the deck. What is the probability they are all the same number? What is the probability only three are the same number? What is the probability that you get two of one number and two of another number?

11. When playing Monopoly, what is the probability of rolling doubles in a turn?

12. In a Monopoly game, you have hotels on Boardwalk and Park Place. Your opponent is on North Carolina Avenue. What is the probability he spends time (and money) at your luxury hotels on the next roll of the dice? On his next turn? For those of you without a Monopoly board handy, Park Place is five squares away from North Carolina Avenue and Boardwalk is seven squares away.

4.4 Poker

Poker is an ideal setting to study probabilities. The game lends models to a variety of different counting techniques and rules of probability. Not to mention it’s fun and the thought of making money off your newly found knowledge of probability is racing through your head. The first step in studying poker is to determine the probability for each hand. There are

\[
\binom{52}{5} = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960
\]

hands and we merely need to count the number of different ways each type of hand can possibly occur.

Previous work has already filled in the four of a kind entry. Let’s tackle the three different yet related flush hands. The royal flush is a 10, J, Q, K, A straight of the same suit. It’s plain to see that there are only 4 such creatures. A straight flush is a straight of the same suit that is not a royal flush. First we must count the number of straights available to us. There exist nine different straights. Recall that we don’t want the 10, J, Q, K, A straight.
<table>
<thead>
<tr>
<th>hands</th>
<th>number of ways</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>royal flush</td>
<td></td>
<td></td>
</tr>
<tr>
<td>straight flush</td>
<td></td>
<td></td>
</tr>
<tr>
<td>four of a kind</td>
<td>624</td>
<td>0.00024</td>
</tr>
<tr>
<td>full house</td>
<td></td>
<td></td>
</tr>
<tr>
<td>flush</td>
<td></td>
<td></td>
</tr>
<tr>
<td>straight</td>
<td></td>
<td></td>
</tr>
<tr>
<td>three of a kind</td>
<td></td>
<td></td>
</tr>
<tr>
<td>two pair</td>
<td></td>
<td></td>
</tr>
<tr>
<td>one pair</td>
<td></td>
<td></td>
</tr>
<tr>
<td>other</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A, 2, 3, 4, 5
2, 3, 4, 5, 6
3, 4, 5, 6, 7

With 9 straights to choose from we next must select a suit. So, we have $9 \times 4 = 36$ straight flush hands. It is critical to note that we have not counted the royal flush in with the straight flush. Not one single poker player would hold a royal flush and seriously call it a straight flush. We must differentiate between the hands as well.

A flush is five cards of the same suit. There are 13 cards of any suit and

$$\binom{13}{5} = \frac{13!}{5! \cdot 8!} = \frac{13 \times 12 \times 11 \times 10 \times 9}{5 \times 4 \times 3 \times 2 \times 1} = 13 \times 11 \times 9 = 1287$$

ways to pick five cards of any one suit. However, our previous work has shown that 9 of those hands are actually a straight flush and one of them is a royal straight flush. Hence, we have $1287 - (9 + 1) = 1277$ hands that are a strict flush for any given suit. Now we multiply by four to consider all possible suits for a flush and get a total of 5108.
<table>
<thead>
<tr>
<th>hands</th>
<th>number of ways</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>royal flush</td>
<td>4</td>
<td>0.00000154</td>
</tr>
<tr>
<td>straight flush</td>
<td>36</td>
<td>0.00015</td>
</tr>
<tr>
<td>four of a kind</td>
<td>624</td>
<td>0.00024</td>
</tr>
<tr>
<td>full house</td>
<td></td>
<td></td>
</tr>
<tr>
<td>flush</td>
<td>5108</td>
<td>0.0019654</td>
</tr>
<tr>
<td>straight</td>
<td></td>
<td></td>
</tr>
<tr>
<td>three of a kind</td>
<td></td>
<td></td>
</tr>
<tr>
<td>two pair</td>
<td></td>
<td></td>
</tr>
<tr>
<td>one pair</td>
<td></td>
<td></td>
</tr>
<tr>
<td>other</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Exercises 4.4

1. Complete the table of poker probabilities.