Proportional (Mis)representation: The Mathematics of Apportionment

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What is apportionment?

Did you ever wonder why Georgia has 14 representatives in the U.S. House of Representatives while Alabama has only 7 and California has 53?

According to the constitution, seats in the House of Representatives should be apportioned among the states "according to their respective numbers".

The constitution when written specified that every state gets at least one seat and the total number should not exceed one for each 30,000 people, but it did not specify how they are apportioned.

- Currently there are 435 representatives (seats) in the House; this number was fixed by law in 1941.
- If a state has $x\%$ of the population, then it should get $x\%$ of 435 seats. But this will almost certainly not be a whole number and we can't assign a fraction of a seat!
- Apportionment is the problem of rounding off sets of numbers so that the sum of the rounded numbers equals the sum of the unrounded numbers.
- Probably the writers of the constitution did not realize that apportionment is a tricky mathematical problem!

Apportionment arises in other contexts:

- A college has 23 different departments and employs 87 people in staff positions. How many staff positions should each department get?
- A mathematics department has 265 students who want to take Calculus I, 147 students who want to take Calculus II, 88 students who want to take Linear Algebra, and 31 students who want to take a course on the Mathematics of Voting. If they can offer 12 sections of mathematics courses, how many of each class should they offer?

We will look at apportionment in the context of giving out seats in the U.S. House, however all definitions and theorems apply to apportionment in general.

The Standard Divisor

- Apportionment is done using census data and historically was finished 2 years after the census.
- The **standard divisor** is the average number of people per seat in the House of Representatives at the time the seats are apportioned.
- Using 2010 census data, the standard divisor is now

$$\frac{308,745,538}{435} \approx 709,760$$

- On average, each representative serves about $709,760$ people.

Apportionment over the years

Here are standard divisors from various years, along with the number of seats Georgia received following apportionment. (The year refers to the census year.)

Year	1790	1850	1900	1960	2000	2010
S.D	33,000	93, 425	194, 182	410, 481	646,952	709,760
GA	2	8	11	10	13	14

The standard quota

- There is one seat for every 709,760 people on average. If each state has population a multiple of 709,760, then apportionment is easy!
- In 2010, the population of Georgia was 9,687,660, which is approximately $13.65 \times 709,760$. Georgia is entitled to 13.65 representatives.
- The **standard quota** (or just **quota**) of a state is the population divided by the standard divisor. For Georgia, this is

$$\frac{9,687,660}{709,760} = \frac{9,687,660}{308,745,538} \times 435 \approx 13.65$$

Apportionment in 1794

- In 1794, when the first apportionment was done, the total population was 3,893,874 with 105 seats apportioned, so the standard divisor was

$$\frac{3,893,874}{105} \approx 37,084$$

- Georgia's population 82,548. So the standard quota of Georgia was

$$\frac{82,548}{37084} = 2.226$$

State	Quota
Connecticut	6.408
Delaware	1.594
Georgia	2.226
Kentucky	1.987
Maryland	8.622
Massachusetts	12.814
New Hampshire	3.826
New Jersey	4.965
New York	9.175
North Carolina	10.651
Pennsylvannia	11.693
Rhode Island	1.864
South Carolina	6.716
Vermont	2.301
Virginia	20.158

Hamilton's Method

In 1794, Hamilton proposed the following method for dividing up the 105 seats:

1. Find the standard quota for each state.
2. Round down each standard quota and give that many seats to the state.
3. Award the remaining seats one at a time to the states whose standard quotas have the largest decimal parts.

President Washington received a bill from congress approving this method and vetoed the bill! It was the first presidential veto in the U.S.

State	Quota	Hamilton's apportionment
Connecticut	6.408	6
Delaware	1.594	1
Georgia	2.226	2
Kentucky	1.987	2
Maryland	8.622	9
Massachusetts	12.814	13
New Hampshire	3.826	4
New Jersey	4.965	5
New York	9.175	9
North Carolina	10.651	11
Pennsylvannia	11.693	12
Rhode Island	1.864	2
South Carolina	6.716	7
Vermont	2.301	2
Virginia	20.158	20

- Nobody knows exactly why Washington vetoed Hamilton's method, but the method has problems.
- Using Hamilton's method, Maryland with decimal part .622 gets an extra seat while Delaware with decimal part .594 does not.
- But Delaware's .594 represents about 37% of its standard quota, while Maryland's .622 is about 7% of its standard quota.
- Delaware has one seat for 59K people, while Maryland has one seat for every 35.5K people.

Jefferson's Method

Jefferson proposed the following method:

- Find the standard quota for each state.
- Round down and if the total number of seats is correct, we are done.
- If not, then choose a new divisor d and calculate a modified quota by dividing each state's population by d.
- Round down the new quotas and if the total number of seats is correct, done.
- Continue with new divisors until done.
- If there are too few seats when rounding down, choose a *smaller* divisor; if too many, choose a *larger* divisor.
- Jefferson's method involves some trial and error, but appears fairer than Hamilton's method.

State	Quota	Hamilton	Jefferson
Connecticut	6.408	6	6
Delaware	1.594	1	1
Georgia	2.226	2	2
Kentucky	1.987	2	2
Maryland	8.622	9	9
Massachusetts	12.814	13	13
New Hampshire	3.826	4	4
New Jersey	4.965	5	5
New York	9.175	9	9
North Carolina	10.651	11	11
Pennsylvannia	11.693	12	12
Rhode Island	1.864	2	1
South Carolina	6.716	7	7
Vermont	2.301	2	2
Virginia	20.158	20	21

- Later, John Quincy Adams proposed a similar method (that was never used). It is the same as Jefferson's method except that in the second step quotas are rounded **up** rather than **down**
- Jefferson's method favors the big states: When rounding down a smaller number, the state loses a larger percent of its quota. Adams' method favors the small states: When rounding up a smaller number, the state gains a larger percent of its quota.
- Jefferson was from Virginia (a large state)! Jefferson's method was used for the first apportionment in 1794. It differed from the apportionment that Hamilton proposed slightly: Virginia gained a seat at the expense of Rhode Island.
- Jefferson's method was used again in 1802 and 1812.

The Quota Rule

In 1822, Jefferson's method was used. New York had standard quota 32.50 and received 34 seats.

An apportionment **satisfies quota** if each state receives a number of seats which is equal to their standard quota rounded up or down. If not, it **violates quota**.

An apportionment method satisfies the **quota rule** if it never violates quota (for any possible apportionment) using the method.

Hamilton's Method satisfies the quota rule. Jefferson's method does not satisfy the quota rule: The apportionment of 1822 violates quota.

Webster's Method

In 1832, the apportionment using Jefferson's method again violated quota.

Daniel Webster was outraged and argued before Congress that giving New York (which had a standard quota of 38.59) 40 seats was unconstitutional.

Webster proposed a method which was the same as Jefferson's except that conventional rounding was used in step 2. In other words, round down if decimal part is $< .5$, otherwise round up.

Webster's method was used in 1842.

Back to Hamilton

- 1852 – There are concerns about Webster's method and quota violations. Congressman Samuel Vinton discovers a wonderful "new" method which is adopted. It turns out to be Hamilton's method!
- 1852– Hamilton's method is used, however Congress deliberately chooses a house size of 234 so that Webster and Hamilton give the same apportionment.
- 1862 – Hamilton's method is used.

Apportionment and the 1876 election

- 1872 – House size of 283 is chosen so that Webster and Hamilton agree. But after much fighting in Congress, it is increased by 9 seats.
- 1872 – Congress passes an apportionment bill not based on any method. This is both illegal (violates the 1852 bill) and unconstitutional (does not use a prescribed method).
- 1876 - Rutherford B. Hayes defeats Samuel Tilden in the presidential election, with electoral votes based on the 1872 apportionment. If Webster's or Hamilton's method had been used in 1872, Tilden would have won the presidency. Mathematics has changed history!!

The Alabama Paradox

At this point, there was no fixed number of seats in the House, instead, Congress looked at apportionment for different house sizes and chose one that seemed seasonable.

In 1882, different apportionment bills were debated and it was noticed that in a House with 299 seats, Alabama would get 8 seats, but in a House with 300 seats, Alabama would get only 7 seats.

The **Alabama paradox** occurs when an increase in the total number of seats apportioned forces a state to lose a seat. In 1882, 325 seats were apportioned using Hamilton's method and there was no paradox.

1902 apportionment and the population paradox

In 1902, the U.S. Census Bureau calculates apportionment using Hamiltons method for house sizes 350-400 seats. Colorado receives 2 for house size 357, but 3 for all other house sizes, so that the Alabama paradox would occur with house size 357.

Rep. A. Hopkins (Illinois) submits a bill proposing a house size of 357! After much fighting, the bill is defeated. Congress chose a house size of 386 and used Webster's method.

It was noticed that if Hamilton's method had been used, then Virginia would have lost a seat to Maine even though the population of Virginia had grown by a larger percentage than Maine. This is an example of the **population paradox**.

The new states paradox

In 1907, Oklahoma became the 46th state and it was decided to add 5 seats to the house and reapportion. However, it was noticed that if Hamilton's method had been used in 1902 and again used for this reapportionment, then New York would have lost a seat to Maine under the new apportionment!

The **new states paradox** occurs if adding a new state and its fair share of seats causes changes in the number of seats awarded to other states.

In 1912, Webster's method was used. In 1922, the 1912 apportionment was carried over, which was unconstitutional! In 1931, a house size of 435 was adopted and Webster's method was used.

Divisor methods

Jefferson's, Adam's, and Webster's methods are examples of **divisor methods**:

Divide each state population by some number – the divisor – and then round up or down in some pre-determined way. Continue with different divisors until the number of seats awarded equals the required total.

It is not too hard to prove that there is always a range of divisors that will produce an apportionment, and there is a unique apportionment.

Theorem

Divisor methods cannot produce any of the three paradoxes described above. Divisor methods never satisfy the quota rule.

Hill's Method

Around 1912, Josepeh Hill, Chief Statistician of the U.S. Census Bureau, proposed a new divisor method, now referred to as Hill's method.

Hill's idea was the following: Give seats out to the states and then check to see if there is any pair of states for which transferring a seat from one to the other makes things more fair, in the sense that the ratio of "people per seat" for the two states is closer to 1. If so, make the switch and look for another pair.

Hill's method was championed by Edward Huntington throughout the 1920's and is sometimes called the Hill-Huntington method.

It turns out the Hill's method is actually a divisor method in disguise!

The **geometric mean** of numbers x and y is $\sqrt{xy}$. Hill's method is simply a divisor method using rounding by geometric mean: Round up if the decimal part is at least the geometric mean of the closest whole numbers, otherwise round down.

For example, the geometric mean of 4 and 5 is $\sqrt{20} \approx 4.472$, so a quota of 4.48 is rounded up, while 4.47 is rounded down.

This has the effect of making it easier for smaller numbers to be rounded up, which makes sense since the decimal part of a smaller number is a larger percentage of the number.

The rest of the story

- In 1941, a committee studied the question of which apportionment method to use and concluded that Hill's method was the best.
- From 1942 - present: the house size is fixed at 435 seats and Hill's method used, both by law.
- In 1992, Montana loses a seat to Washington and challenges the constitutionality of Hill's method. The Supreme Court upholds the method.
- In theory, Hill's method could violate quota, however if Hill's method had been used for all apportionments (1794-2002), they would all satisfy quota.

Problems with apportionment methods

- Any of the divisor methods can violate quota.
- Hamilton's method will never violate quota.
- All of the apportionment paradoxes can occur with Hamilton's method.
- The apportionment paradoxes cannot occur with divisor methods.
- All of these methods satisfy **monotonicity**: No state receives fewer seats than a state with with less (or the same) population.
- Since both Hamilton and divisor methods have problems, is there a better method?

An example

The mystical land of Psykozia is divided into four states: Bliss, Confusion, Disarray, and Ignorance.

There are 7 seats in the House of Representatives, and they are to be apportioned among the four states. Let b be the number of seats Bliss gets, c the number of seats Confusion gets, etc.

Assume that we want to use an apportionment system which satisfies monotonicity and the quota rule.

An example, cont.

Populations of the states and the corresponding standard quotas are as follows:

State	Population	Quota
Bliss	3003	5.005
Confusion	400	0.667
Disarray	399	0.665
Ignorance	398	0.663

By monotonicity, $b \geq c \geq d \geq i$. By the quota rule, the only possible apportionments are $5, 1, 1, 0$ or $6, 1, 0, 0$, so Bliss gets at least 5 seats and Ignorance gets 0 seats.

An example, cont.

At the next census, the total population has increased by 1100, the new numbers are:

State	Population	Quota
Bliss	3004 (+1)	3.968
Confusion	1503 (+1103)	1.985
Disarray	396 (−3)	0.523
Ignorance	397 (−1)	0.524

It is easy to see that by monotonicity and the quota property, the only possible apportionments are $(4, 2, 0, 1)$, $(4, 1, 1, 1)$, or $(3, 2, 1, 1)$. But this means that Bliss has gained population and lost a set, while Ignorance has lost population and gained a seat. Therefore, the population paradox has occurred.

We have just proven a famous impossibility theorem from 1980:

Balinski-Young Theorem

There is no apportionment method that satisfies monotonicity and the quota rule, for which the population paradox never occurs.

After proving the theorem, Balinski and Young went on to give an argument that Webster's method is the best. In fact, many apportionment experts consider Webster's method to be better than Hill's.

There are many, many mathematical questions to be asked about apportionment, and research continues today.

Further Reading

The Mathematics of Voting and Elections: A Hands-On Approach, J. Hodge and R. Klima, American Mathematical Society (2005).

Mathematics and Politics: Strategy, Voting, Power and Proof (2nd edition), A. Taylor and A. Pacelli, Springer (2008)

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Fair Representation, M. Balinski and H. Young, Yale University Press (1982).